THREE HUNDRED YEARS OF MATHEMATICS AT HARVARD

J. L. COOLIDGE, Harvard University

1. The first century. The beginning of mathematical study at Harvard was contemporaneous with the beginning of the college, and both beginnings were indeed small. It was on October 28, 1636 that the Great and General Court of Massachusetts "agreed to give 400 £ towards a schoale or colledge." The first classes seem to have been held in July, 1638, but the real founding was in 1640 when an unusual man, Henry Dunster, assumed the duties of the presidency. The grant of 400 £ was generous, and so was the bequest of John Harvard who, at his death in 1638, left one half of his estate, £779 17s 2d, and all of his library to the infant institution. However, the need for additional funds was immediately evident. The most likely source for such aid was that body of people in Old England who were deeply interested in the creation of a Commonwealth in New England, which should conform to the principles of the Puritan creed. Accordingly in 1643 an appeal was addressed to them entitled "*New England's First Fruits.*" The object of this was ostensibly to set forth what had been accomplished in Massachusetts Bay. The tone is pious and edifying; not so the comment in Morison,* " '*New England's First Fruits*' was a promotion pamphlet; one half expects to find in it a return post card on receipt of which our representative will call." But the document pays a good deal of attention to the college, and we read that the faculty, that is to say President Dunster, at 10:00 A.M. on Mondays and Tuesdays read arithmetic and geometry to the students of the third (and last) year during the first three quarters, and astronomy in the last quarter. And so mathematics began in our oldest college.

It is probable that there was little change in the next eighty years. By 1653 mathematical theses appeared on the Commencement programme. Dunster taught geometry in English, though Latin was the language of the college, basing his teaching on Euclid and Ramus. In astronomy, surveying and navigation, progress was more marked as we see in Morison.†

With the beginning of the eighteenth century some progress had been made. In a thesis of 1711 there is mention of the conic sections, in 1719 there is a reference to fluxions, and in 1721 we find algebra. These theses were apparently statements which the candidate was prepared to defend. "A fluxion is the velocity of an increasing or decreasing flowing quantity." "Algebra is the art of reasoning with unknown quantities in order to define their relation to known quantities." There must have been enough substance to all this to stimulate that unusual student to whom I now turn.

2. Isaac Greenwood. The greatest step forward came in 1726 with the appointment of the first mathematical professor, Isaac Greenwood. There lived in

* Samuel Eliot Morison, The Founding of Harvard College, Cambridge, 1935, p. 304.

† Harvard in the Seventeenth Century, Chapter X.

London at that time a man named Thomas Hollis who was deeply interested in Harvard for reasons that are not now evident. In 1721 he established a professorship of Divinity which the college proceeded to accept, and then to violate the spirit of the gift.* He planned to establish a professorship of mathematics and natural philosophy by his will, but meeting Greenwood in London changed his mind and gave the sum, £1200, outright, suggesting that Greenwood be appointed.

Isaac Greenwood graduated from Harvard in 1721, and received an A.M. degree three years later. He then went to London, studied divinity and preached, but I judge did some mathematics also. At any rate he somehow acquired more knowledge of that subject than was easily attainable in Cambridge, Massachusetts. For a time he fascinated the good Hollis, but doubts crept in for we find Hollis writing to the Harvard Corporation in July 1726,† "I advise you to make due trial of him for your own satisfaction. He has not pleased me of late. Only you may know that if you recommend I accept, not else." Some little time later he writes in the same vein, "Mr. Greenwood has left us on a sudden without paying his debts, or taking leave of his landlord, his tutor or me;" and again that the money Greenwood had spent in a ramble of a few weeks and his debts to his tradesmen and others amounted to three hundred pounds sterling. Among other instances of extravagance were three pairs of pearl colored stockings.

One would naturally expect that the Harvard authorities would hesitate to appoint a professor with such a record. Perhaps they did hesitate, but they surely appointed him "under the strong apprehension, probably entertained by both boards, that if they hesitated a Baptist might be forced on them."‡

Hollis's fears proved well grounded. Quincy tells us "The proceedings in respect to this unhappy individual were marked with consideration and firmness. On the 26th of April, 1737 Mr. Greenwood was called before the Overseers charged with intemperance which he confessed, and casting himself on the lenity of the Board professed his resolution to reform. Several lapses and warnings followed. On November 25 he was required to exhibit a humble confession and received a public warning. He lapsed twice between then and December 7 when a formal vote of expulsion was passed, but six months of further repentance and lapses followed till he finally left on July 13, 1738.§

So much for Greenwood the citizen. As a mathematician he was a decided success. He was interested and wide awake, publishing in 1726 a course of philosophical lectures explaining Newton's theories and in the same year a "*Course of mechanical philosophy.*" In 1729 appeared his "*Arithmetic vulgar and decimal,*" the earliest work on that subject by an English speaking American. He also offered to give private instruction in the novel subject of flux-

* Quincy, History of Harvard University, Cambridge, 1840, Vol. I, pp. 238 ff.
† Quincy, Vol. II, pp. 11 ff.
‡ Quincy, Vol. II, p. 21.
§ Quincy, Vol. I, pp. 11 ff.

ions. We get a close view of Greenwood's influence on his pupils from the manuscript notes of two of them which have been recently published in Simons.* These are the notebooks of Samuel Langdon, subsequently President of the university, and John Diman. The subject is algebra. They cover all of the usual present school topics, with quite a little attention to cubic equations, and a short account of approximate determination of the roots of equations of any order. There are also some geometrical problems solved by algebraic manipulation. There are references to the works of Raphson, Oughtred and Halley, probably accessible to Greenwood, but not to his pupils.

The College might well have been embarrassed to find an adequate successor to Greenwood had not Fate kindly provided a thoroughly capable one in the person of John Winthrop, who served as Hollis's professor for forty-one years. He has been called the father of American astronomy, and did notable work in that science, being a Fellow of the Royal Society, and contributing to their Transactions. He had a wide interest in various branches of science, among them physics, astronomy and seismology. I can not find that his interest in pure mathematics was outstanding, but between 1735 and 1750 certain higher plane curves appear in the Harvard mathematical theses, and after 1751 fluxions occupied a leading place.†

3. The first decline. The decline in the eminence of the holder of the Hollis Professorship, which was fortunately averted by the appointment of Winthrop, finally set in at his departure. Samuel Williams next took the chair. I can not find out much about him, and little more concerning his successor Samuel Webber. It is true that the latter published two or three volumes of mathematical textbooks "taken from the best authors" and managed to be elected president of the College. Cajori‡ quotes Edward Everett as saying that Webber was a person of tradition and routine, and Story that he was modest, mild and quiet, but unconquerably reserved and staid, but Quincy is more eulogistic: "The urbanity and gentleness of his manners, the prudence, rectitude and firmness which characterized his service in the office of president, secured for his administration popularity and success, both with his pupils and with the public.§ There was certainly a retrocession in the interest in mathematics during these years. We have seen how much attention was paid to fluxions during the latter part of the eighteenth century, but Cajori‖ tells us that of 133 mathematical theses written between 1781 and 1807 only seven involved fluxional problems, though much attention was paid to astronomy. In 1803 John Farrar wrote a "*Calculation and Projection of a Solar Eclipse*," quite an advance on Samuel Farrar who

* Lao Ginevra Simons, "Introduction of Algebra into American Schools in the Eighteenth Century," U.S. Bureau of Education, 1924. No. 18.

† Simons, p. 38.

‡ Cajori, "The Teaching and History of Mathematics in the United States," Bureau of Education Circular No. 3, 1891.

§ Quincy, *loc. cit.*, Vol. II, p. 299.

‖ Cajori, *cit.*, p. 59.

in 1793 offered "*A Prospective View of the Episcopal Church.*"

This same John Farrar next claims our attention, for he succeeded to the Hollis Professorship. One can find differing opinions of him. Cajori* quotes the excellent Andrew P. Peabody as writing, "He delivered when I was in college a lecture every week to the junior class on natural philosophy and one to the senior class on astronomy. No student was willingly absent . . . he very soon rose from prosaic details to general laws which he seemed to ever approach with blended enthusiasm and reverence as if he were investigating and expounding divine mysteries. His face glowed with the inspiration of the theme." I quote also from Smith-Ginsburg:† "He was one of the best teachers of his time," but Cajori quotes Josiah Quincy as saying "We gained a miss from Farrar for the fourth time this term. This was much to the gratification of the class, who hate his branch, though like him." Whether Farrar was a good teacher or not, he knew good teaching when he saw it, for he published translations of excellent French texts by Bezout, Lacroix and others.

4. Benjamin Peirce. The first epoch in the history of mathematics at Harvard opened with the opening of the college, the second, nearly a hundred years later, with the appointment of Greenwood. The third epoch began about a century later still with the appointment of Benjamin Peirce in 1832. A long account of him will be found in Cajori‡ and "*Benjamin Peirce*" by R. C. Archibald.§ Here we have a man in quite a different class from any of his predecessors, and able to bear comparison with any who have succeeded him. He had a brilliant and very rapid mind, a profound interest in science, and a conception of a university as a place where mathematics should not only be pursued but advanced, both by teachers and pupils, which was quite different from anything in evidence in America before his time. As a scientist his most important work was in astronomy. In mathematics his outstanding work was his "*Linear Associative Algebra.*" This highly original treatise has been sharply attacked, perhaps without sufficient reason. A careful discussion will be found in Archibald. His membership in the Royal Society of London, Royal Astronomical Society, Royal Society of Edinburgh, Royal Society of Göttingen and Honorary Fellowship in the University of Kiev show the esteem in which he was held outside his own country. A complete list of his honors is found on pages 10 and 11 of Archibald. In originality and power no contemporary American mathematician except Hill and Gibbs, if they deserve that classification, was comparable to him.

There is no unanimity of opinion of Peirce as a teacher. I record the fact that the late A. Lawrence Lowell who studied mathematics under Peirce, writing a graduation thesis on Quaternions, expressed the view that the most stimulating teachers he ever met were Henry Adams in history, and Peirce in mathe-

* Cajori, *cit.*, p. 127.

† "A History of Mathematics in America before 1900," by David Eugene Smith and Jekuthiel Ginsburg, Carus Monograph No. 5, Open Court Publishing Co., 1934, p. 97.

‡ Cajori, *cit.*, pp. 132–147.

§ This MONTHLY, January 1925, and separately Open Court Publishing Company.

matics. Now Lowell attended the Harvard Law School in the days of Langdell and Ames, of Gray and Thayer. Cajori on page 143 quotes Thomas Wentworth Higginson as saying, "We did not know whither he was going, but the huddle of new equations seemed like an outlet from the World, and a ladder to the stars. He gave a charm to the study of mathematics which, for me, has never failed." Many able astronomers studied with him, G. P. Bond, Asaph Hall, B. A. Gould, Simon Newcomb, and G. W. Hill. But there is little doubt that he was a sad failure as a teacher for ordinary students. In 1838 elaborate plans were made for freshman and sophomore mathematics at Harvard, but Cajori remarks on page 136, "This arrangement did not prove satisfactory. Professor Peirce's textbooks were found very difficult, and Peirce himself was not a good teacher except for boys of mathematical genius." In 1848 the Lawrence Scientific School was opened at Harvard, and Peirce presented a wonderful array of courses in mathematics and astronomy, reading the works of Cauchy, Lagrange, Laplace, Gauss and others. What a marvelous programme for the United States in 1848! But there were only two takers in 1849, and all was soon dropped. Repeated and loud complaints were made that mathematical teaching at Harvard was poor. Cajori quotes W. F. Allen of the class of 1851 as saying that only two members of his class of sixty elected mathematics, and they soon dropped it.

And what of Peirce's textbooks? Cajori says on page 441 that Peirce's *Analytical Mechanics* was acknowledged at the time, even in Germany, to be the best of its kind, but Archibald on page 14 quotes Newcomb as writing "The exposition of dynamical concepts in the first forty pages is pleasant reading for one already acquainted with the subject but that the student beginning the subject could understand it without clearer distinction of definitions, axioms and theorems, seems hardly possible." In 1848 a committee of the Overseers reported that Peirce's textbooks were abstruse and difficult and few could comprehend them without explanations, that his work was symmetrical and elegant, and could be pursued with pleasure by adult minds but that books for young students should be more simple. A minority report, submitted by Thomas Hill and J. Gill, disagreed totally, and commended the works for their terseness, simplicity of style, vigor and originality of thought.*

It is true that Peirce's books are brief and original. I have the impression that this conciseness was often attained by neglecting the real difficulties involved. Says Cajori on page 144, "As an instance of this we mention his assumption as self evident that a line which is contained upon a limited surface, but has neither beginning nor end on the surface, must be a curve reentering on itself. By this new axiom he reduces a demonstration which would occupy half a dozen pages, to a few lines!" This is certainly neat, but unfortunately the axiom, even if self-evident, is not always true. One has but to consider a double spiral which cuts at an angle, other than 90°, all the circles through two points. Turn to his *Curves, Functions and Processes*, Vol. I, page 176.

* Cajori, pp. 140–141.

To find the nth power of $1+i$, when i is infinitely small and n infinitely great so that $ni=a$

$$(1+i)^n = 1 + ni + \frac{n(n-1)}{1\cdot 2} i^2 + \frac{n(n-1)(n-2)}{1\cdot 2} i^3 + \cdots .$$

But n is infinite, therefore

$$n-1 = n, \qquad n-2 = n, \text{ and } \cdots .$$

$$(1+i)^n = 1 + ni + \frac{n^2 i^2}{1\cdot 2} + \frac{n^3 i^3}{1\cdot 2\cdot 3} = 1 + a + \frac{a^2}{1\cdot 2} + \cdots .$$

Put $a=1$

$$(1+i)^{1/i} = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdots$$

$$= (1+i)^{a/i} = [(1+i)^{1/i}]^a = e^a = 1 + a + \frac{a^2}{2} + \cdots .$$

$$1 + i = e^i; \qquad \log (1+i) = i.$$

If $a=mi$

$$(1+i)^m - 1 = a.$$

"All terms of the second member except the first being neglected because they are infinitely small."

If $e^m=b$

$$b^i - 1 = i \log b.$$

To differentiate a^x

$$da^x = a^x(a-1) = \log a \cdot a^x dx.$$

No wonder Peirce's work is condensed.

I have frequently wondered why it was that Peirce who was a close student of Cauchy and Gauss should be so completely oblivious of rigor. His pupil, Byerly, once told me that the master never bothered whether a series were convergent or not. The only explanation is that such things must have seemed to him unimportant and not worth caring about. Perhaps an explanation is also found in the following anecdote. I quote from Charles W. Eliot in Archibald, page 2, "One day in my senior year I ventured to say that what he had just been saying to us about functions and infinitesimal variables seemed to me to be theories and imaginations, rather than facts or realities. Peirce looked at me gravely and remarked gently, 'Eliot, your trouble is that your mind has a skeptical turn.' " Such a reproach was never addressed to Benjamin Peirce himself.

5. The period of retrocession. With the retirement of Benjamin Peirce in 1878 there came a great slump in the scientific activity of the Harvard mathematical department, although the teaching continued and improved. His son James Mills Peirce was a much better teacher, even though he lacked the spark of originality; his contributions to mathematics are negligible. Like his father he taught for forty-nine years, a most kindly and genial man. Like his father he was deeply interested in Hamilton's "Calculus of Quaternions", and lectured on the subject regularly until his death in 1905. He deserved well of the University, for his pioneer work in introducing the elective system, and building up the Graduate School.

Many Harvard men will recollect William Elwood Byerly of the class of 1871, who took the first Harvard degree of Ph.D. in 1873. He spent the next three years teaching at Cornell, and fell under the influence of Evan W. Evans who had lectured on modern methods in analytic geometry, broadly speaking, the sort of thing one finds in Salmon's Conic Sections. When Byerly came back to teach at Harvard in 1876, he offered a course dealing with these matters which became standard in Harvard teaching for the next sixty years. He was a remarkable elementary teacher, and his two textbooks on the "*Differential and the Integral Calculus*" marked a real advance. He had been brought up on Peirce's texts and when he encountered the best French books, notably the work of Bertrand, a great light broke on him, and he determined to publish something similar. It is hard to say what he might have accomplished had he been less burdened with teaching at Harvard College and administrative work at Radcliffe.

About contemporary with Byerly was the third Peirce, Benjamin Osgood, a singularly retiring and modest man, but a thorough scientist. He tied mathematics in with physics in a way that has been frequently lacking at Harvard especially from the time of his death in 1915 to the appointment of J. H. VanVleck in 1934.

6. The new movement. The renaissance of mathematics at Harvard came in the early nineties with the appointment of the "great twin brethren" William Fogg Osgood and Maxime Bôcher. It was the moment of the great awakening in American mathematics, when a number of able and enthusiastic young men, largely trained in Germany, set about raising the science as pursued in this country to the same plane on which it was pursued in Europe. The immediate means was the American Mathematical Society with its Bulletin and Transactions. It was the beginning of the era of Moore, Maschke and Bolza in Chicago, of Fiske and Cole at Columbia, of Fine at Princeton, Pierpont at Yale, VanVleck at Wisconsin and White at Vassar. Osgood and Bôcher, early presidents of the Society, were in the very middle of it. They introduced into Harvard new and advanced courses, largely dealing with what one might call "Göttingen mathematics," gathered around them graduate students whom they prepared for the doctorate, and treated their younger colleagues who came after them with

the greatest sympathy and kindness. Both were good teachers for curiously different reasons. Osgood was extremely conscientious. He felt it was his business to teach and teach well, to make his pupils understand. So he worked over a subject, turning it about in every way till it assumed a shape which he believed a beginner, whether a freshman or graduate student, could understand. Bôcher was different. His father was French and he had a limpid Gallic mind. He taught clearly because he thought clearly, that was the way his mind worked. He was cut off by death in 1918. Osgood continued until his retirement in 1933, the first member of the Department to retire quietly at the natural time since Benjamin Peirce withdrew fifty-five years before. As a sign of the recognition of these men in Europe it is worth noting that Bôcher wrote the article on "*Randwertheaufgaben*" in the Enzyklopädie, Osgood the even more important one on "*Analytische Funktionen.*" His "*Lehrbuch der Funktionentheorie*" is surely the most important of the eighty mathematical books which the members of the Department have published since Greenwood's appointment in 1726.

In the third and fourth decades of the present century the dominant figure in Harvard mathematics has been George David Birkhoff. His preference has been for substantial classical mathematics, but he has touched a variety of topics in turn. He was early interested in Poincaré's papers on celestial mechanics; the three body problem struck his fancy, as well as the stability of orbits. He was thus naturally led to the geometrical problem which Poincaré tried hard to solve and finally abandoned. Birkhoff solved it in 1912. His work in dynamical systems has been fundamental, and even more could be said of his solution of the ergodic problem. One is naturally tempted to compare Birkhoff with Benjamin Peirce, but there is no fair ground of comparison, for the conditions of American mathematics have changed radically in the interval. Moreover, when Peirce published, his were the only Harvard contributions to mathematical science. In Birkhoff's time scientific activity was very general throughout the Department. It would be a great mistake to pass over in silence the work of Morse and Stone, to mention but two of many.

7. The Lowell epoch. Important changes in the work of the Department came during the twenty-three years' presidency of Lowell, 1909–1932. There had been for a number of years two kinds of introductory mathematical instruction, one in the College, the other in the Lawrence Scientific School. The distinguished pupil of Benjamin Peirce could not recognize the existence of two different kinds of mathematics; then why have two kinds of instruction? The Department held the same opinion, but insisted that in taking on an additional burden of elementary teaching they should not be handicapped by the addition of new permanent members who were capable only in the elementary field. The University authorities accepted this view. E. V. Huntington came over with the Scientific School courses; the appointments of Birkhoff and Dunham Jackson were more or less connected with these changes.

In 1913 the visiting committee of the Board of Overseers found that the freshman sections were too large, and recommended that smaller sections would

be better for the majority of beginners. They did not suggest that the burden of elementary instruction falling on the permanent members of the staff be correspondingly raised. The solution seemed to be to give temporary half-time appointments to able young men who were deemed capable of handling sections of fifteen or twenty men. The solution was not ideal. It was not always easy to find a sufficient number of mathematical aspirants who had not only the mathematical knowledge, but the skill and personality. Their work was closely supervised by the permanent members, and the system has, all things considered, worked well. What better plan could be found for teaching four or five hundred freshmen yet maintaining standards in the higher branches?

A fundamental change in the Harvard requirements for the A.B. degree was introduced gradually during the Lowell regime. This was the introduction of a final general examination in a subject, not a course. The preparation for this involved the appointment of a body of tutors. The Department of Mathematics took some time to fall in with this plan, but finally accepted it in 1926. They insisted, however, that tutoring should be considered as one of the functions to be assumed by all of the staff, and not the specialty of a separate and somewhat nondescript body of teachers. Each permanent member should give a definite part of his time to tutoring; this usually came to the responsibility for five pupils. Once again they insisted, however, that they should only undertake this if they were strengthened accordingly. The authorities agreed. Marston Morse and Marshall Stone were appointed.

The members of the Department have taken part in various mathematical activities outside of the University. They have held many offices in the Mathematical Society and the Mathematical Association. Many have done editorial work on mathematical journals. For a number of years beginning in 1899 they took over from the University of Virginia the responsibility for publishing the Annals of Mathematics. The subsidy to this from the University was subsequently withdrawn, and the editorial responsibility for publishing passed to Princeton. The Harvard Department felt that they could do more for mathematical science by giving their time and strength to their own research and training their own pupils than by editing the writings of others. This may have been narrow and shortsighted. The mathematical future was not easy to predict at that time.

8. Fundamental principles. In the last seventy-five years the Harvard Department has evolved a definite doctrine of its duties and responsibilities. These fall under four heads:

1. To inject the elements of mathematical knowledge into a large number of frequently ill informed pupils, the numbers running up to 500 each year. Mathematical knowledge for these people has come to mean more and more the calculus.

2. To provide a large body of instruction in the standard topics for a College degree in mathematics. In practice this is the one of the four which it is hardest to maintain.

3. To prepare a number of really advanced students to take the doctor's degree, and become university teachers and productive scholars. The number of these men slowly increased from one in two or three years, to three or four a year.

4. To contribute fruitfully to mathematical science by individual research.

To fulfill these tasks the Department gradually evolved one sacrosanct and fundamental rule; all permanent members must have the same duties and responsibilities. I think that this conception goes back to J. M. Peirce. It seems somehow characteristic of his generous nature. Perhaps the oldest members receive favored treatment in the choice of hours, otherwise there is no distinction. Each must do his share of tutoring, of elementary teaching, of standard teaching, of dull administrative detail. Each must be ready to help any qualified advanced pupil who wishes to work with him. Of course, each must do his share of research, not only do it, but want to do it. It can not be maintained that all are equally good teachers of elementary courses. But the Department has always been skeptical about the man who is merely a teacher, especially an elementary teacher, and nothing else. Such a man, freed from competition and lacking the stimulation of original work, is often inclined to become an unprogressive and stereotyped teacher. As for research, no one imagines that the work of all is equally important scientifically, but it is morally. There is a diversity of gifts, but the same spirit.

The fine fruit of all of this has been the feeling, frequently shown, that what counted was the success of the combined undertaking, not the triumphs of an individual. On several occasions a senior man, who thoroughly deserved any advancement the University could give, has welcomed the promotion over his head of a junior man when that seemed for the advantage of the whole body. I do not think that any set of teachers, no matter how brilliant, who were essentially career men, could have accomplished what has been done by the Harvard Department of Mathematics.

The Derivatives of Composite Functions

Since the publication of my note on "The derivatives of composite functions" (p. 9 of the present volume of this MONTHLY), my attention has been called to a number of papers on this and related topics. Among these I should mention a paper by L. S. Dederick, "Successive derivatives of a function of several functions," Annals of Mathematics, vol. 27, 1925–26, p. 385 and one by I. Opatowski, "Combinatoric interpretation of a formula for the nth derivative of a function of a function," Bulletin of the Am. Math. Soc., vol. 45, 1937, p. 944. Each of these gives references to further interesting papers.

ARNOLD DRESDEN

MATHEMATICS, 600 B.C.–400 B.C.*

MAX DEHN, Illinois Institute of Technology

1. Isolated arithmetical and geometrical facts were, without doubt, known in prehistoric times much as such facts are now known among the most primitive tribes. Rather advanced mathematical knowledge appears in ancient Egyptian papyri (for instance in the Rhind Papyrus of the 14th century B.C.) and on numerous Babylonian cuneiform texts dating from 2000 B.C. onwards. Certainly the Greeks learned many of the algebraic methods and the techniques of geometric measurements from these ancient peoples through the lively commerce of the Eastern Mediterranean. Our reports begin with Greek mathematics after 600 B.C.

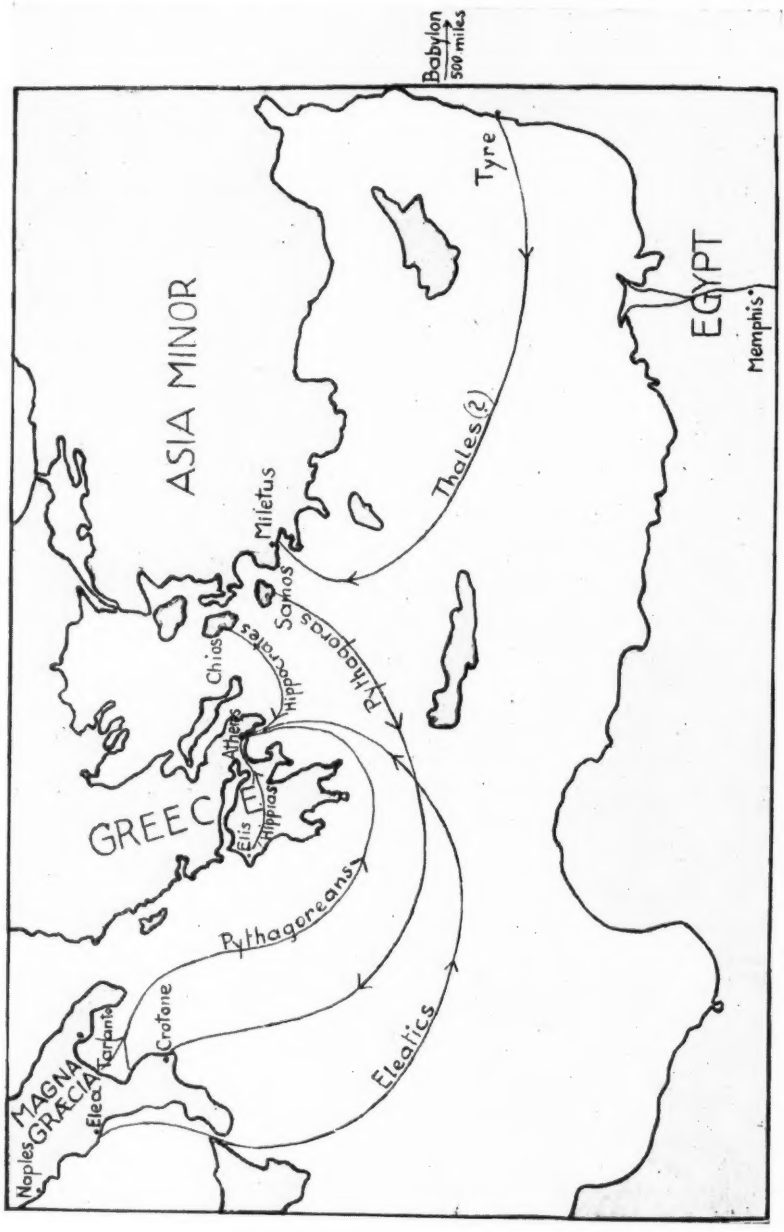
2. Sources. The Sources for the history of Mathematics in Greece during the period from 600 B.C. to 400 B.C. are very scarce and unreliable. We have a fragment of mathematical history by *Eudemos* (ca. 320 B.C.) in an excerpt of the sixth century A.D. This fragment itself is in a bad state, corrupted by later changes. There are, however, scattered among the works of Greek authors, enough passages concerned with the mathematics and mathematicians of Ancient Greece, for us to derive a fairly clear idea of this early period.

3. Early Greeks. While there is no mathematician known from ancient Egypt or Babylon, we do know the names of famous Greek mathematicians. *Thales* (ca. 600 B.C.) of Miletus (see map), who was probably of Phoenician origin, is known as the father of Greek mathematics. He had many disciples. It may be that there is a direct connection between him and *Pythagoras* (ca. 550 B.C.) from Samos. The latter, who was the head of an aristocratic brotherhood, a school of wisdom and science, was a political and philosophical leader in Southern Italy. He emphasized the importance of Mathematics in the higher or liberal education, and for many centuries his name invoked an aura of mysticism. After his death, his school flourished for more than a hundred years, and numbered several famous mathematicians among its members. They will be mentioned in the second report.

Hippocrates of Chios (ca. 450 B.C.) was probably not connected with the Pythagoreans. He taught mathematics at Athens. We have a fragment of his mathematical work transmitted by Eudemos. This is the first *published* mathematical investigation known. Hippocrates is probably also the author of the first manual of geometry.

Hippias of Elis (ca. 430 B.C.) was a famous Sophist, a man with vast knowledge in mathematics and astronomy. An outstanding teacher, he was paid for his courses, which he gave mainly at Athens, where teaching and research in

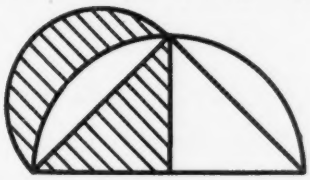
* This is the first of a series of articles by Professor Dehn which, it is hoped, will cover the whole history of mathematics in compact form. An important feature of the series is the collection of maps, which are the work of Mrs. Dehn. Professor M. H. Heins, of Illinois Institute of Technology, is doing some of the editorial work connected with the articles.


Babylon
500 miles
Tyre
EGYPT
Memphis
ASIA MINOR
Miletus
Thales(?)
Samos
Chios
Hippocrates
Pythagoras
Athens
GREECE
Elis
Hippias
Pythagoreans
Eleatics
Crotone
Tarentum
Elea
MAGNA GRACIA
Naples

mathematics were concentrated at the end of the period with which the present report is concerned.

Even in this early period we begin to see many of the features of modern scientific activity: authors famous for their achievements, ambitious to find new results; renowned teachers; pupils eager to learn; books where results are collected, digested, and presented in such a way that the reader understands the facts and proofs, and is inspired to do research himself.

4. Achievements. What is left to us of Babylonian and Egyptian mathematics shows only prescriptions for computations or for solutions of particular problems. But we find in the old Greek mathematics, proofs of the given solutions of problems and of the various theorems; we find convincing explanations. Great problems are proposed and treated. Problems of construction are solved with the help of ruler and compass. Among such problems are the conversion of areas into each other, the most important case being the squaring of the rectangle; and the construction of the regular pentagon by means of the golden ratio.



Also propounded at this period were three classical problems of construction: the squaring of the circle, the trisection of an angle, and the duplication of the cube. The first two problems stimulated the *construction of the first curve* apart from the "naturally" given circle. This curve, which was invented by Hippias, is the quadratrix, whose equation in rectangular coordinates is $y=x \cot \pi x/2r$. The construction of points on this curve, approaching the y-axis at the level $y=2r/\pi$, corresponds to the Archimedian computation of the perimeters of regular polygons approaching the circumference of a circle. In the construction of Hippias, the limiting process is visualized by the continuous curve approaching the y-axis.

The problems of trisecting an angle and of duplicating the cube led to new mechanical devices other than the compass, and finally, at the beginning of the next period, resulted in the discovery of the conics.

5. Theorems. Probably the first theorems, found by the Greeks, were propositions about angles. The *Pythagorean theorem*, as a relation between the lengths of the sides of a right triangle, was in all likelihood already known to the Babylonians; as a theorem about areas it is perhaps a Greek achievement. At all events, the knowledge of this theorem which was always attributed to Pythagoras himself, was a matter of great moment to all educated Greeks.

Endeavoring to square the circle, Hippocrates discovered areas bounded by two circular arcs which could be constructed by compass and rule. The simplest

case is indicated in the preceding figure, in which the two shaded areas are equal.

To this period belongs the discovery and the construction of the *five regular solids*.

The greatest achievement of this epoch was the discovery and proof of the existence of *irrational ratios* in the incommensurability of side and diagonal of a square. Whether the original proof was given by arithmetical or geometrical methods is unknown. This was the first example of a mathematical truth contrary to naïvely simplifying intuition. The necessity for a strict proof became apparent, and this influenced the whole development of mathematics in the direction of rigor.

Further, we have the discovery of the projection of the infinite process of counting into arithmetical, geometrical, and kinematic ideas. These phenomena were found and discussed by the *Eleates* (Elea, a city of Southern Italy). The finite sum of an infinite geometric progression, the indefinite subdivision of a finite line or of a finite movement were all in contradiction to naïve intuition and provoked profound problems as well as new constructions in Philosophy.

WHAT IS A MATRIX?

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1. History. J. J. Sylvester, who had a penchant for startling nomenclature, introduced the term *matrix* into mathematics [1]. He applied it to a rectangular array of numbers "out of which determinants can be formed." Two such arrays were considered to be equal only if corresponding elements were equal. Arthur Cayley [2] used the term in the same sense, as any array of coefficients, with the insistence that "the idea of matrix precedes that of determinant."

The branch of mathematics which is now called the algebra of matrices had four sources. W. R. Hamilton [3] first presented it under the title of "Linear and vector functions." Cayley [4] considered a matrix as a single quantity, defined sum, product and scalar product, and stated that "square matrices are of greater importance than rectangular." E. Laguerre [5] also laid the foundations of matric algebra, and remarked that "the calculus of matrices gives a simple interpretation of ordinary complex numbers, of quaternions, of the algebraic clefs of Cauchy, of the imaginaries of Galois." G. Frobenius [6] encountered matric algebra in the guise of bilinear form theory. In the awkward notation of the composition of forms, he discovered many of the fundamental theorems of matric theory. The abstract point of view of Cayley, and Sylvester's term *matrix*, gradually became standard.

2. The total matric algebra. If the universe consisted of a single particle of matter, the phenomenon of gravitation could not exist. In fact, the particle would be practically without properties, for the physical properties of matter can for the most part be described only in terms of reactions among two or more

particles. A single particle of matter would be a forlorn and uninteresting thing.

Similarly a mathematical entity, such as number, point or element, is a name and nothing more until its properties are defined; and its properties involve its relations with the other entities of the mathematical system to which it belongs. For instance, it is futile to attempt to define a complex number in any other way than as an element of the complex field. The complex field can be defined by giving enough of its properties to characterize it, whereupon a complex number is set apart from every other kind of number by its properties which arise from its membership in the complex field.

To define a matrix as a rectangular array of numbers is as inadequate as to define a single particle of matter or a single complex number. *A matrix is a number of a total matric algebra.*

A total matric algebra of order n over a ring R[7] may be defined as follows: Consider the set of all $n \times n$ arrays of elements of R such as

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \cdots b_{1n} \\ b_{21} & b_{22} \cdots b_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \\ b_{n1} & b_{n2} \cdots b_{nn} \end{bmatrix}, \cdots .$$

More briefly we may write $A = (a_{rs})$ where r is the row index and s the column index of the element a_{rs}. We shall understand that $A = B$ means

$$a_{rs} = b_{rs} \qquad r, s = 1, 2, \cdots, n.$$

Three operations, addition $(+)$, scalar multiplication $(\cdot)$ and multiplication $(\times)$ are defined by the identities

$$A + B = (a_{rs} + b_{rs}), \qquad k \cdot A = (ka_{rs}),$$

$$A \times B = \left(\sum_{i=1}^{n} a_{ri} b_{is} \right)$$

where k is any number of R. It may readily be verified that addition is associative and commutative, and that the array O all of whose elements are 0 is such that

$$A + O = O + A = A$$

for every A. The array $-A = (-a_{rs})$ is such that

$$A + -A = O.$$

Thus the arrays of M form a commutative group with respect to addition. Multiplication is associative, and the following distributive laws hold:

$$A \times (B + C) = A \times B + A \times C, \qquad k \cdot (A + B) = k \cdot A + k \cdot B,$$

$$(k + l) \cdot A = k \cdot A + l \cdot A.$$

The mathematical system M so defined is called the *total matric algebra* of order n^2 over R, and the arrays $A, B, \cdots$ are called *matrices* of order n over R.

Since R is a ring, M is a ring. If R has a unit element 1, M has the unit element

$$I = (\delta_{rs}), \qquad \delta_{rr} = 1, \ \delta_{rs} = 0 \text{ for } r \neq s,$$

and the scalar multiples of I constitute a subalgebra of M isomorphic with R. But the assumption of the commutativity of R does not imply the commutativity of M for $n>1$.

Let R be a ring with unit element 1, and let ϵ_{ij} be the matrix whose element in row i and column j is 1, and all of whose other elements are 0. Then

$$A = \sum_{i,j=1}^{n} a_{ij}\epsilon_{ij}$$

so that M may be thought of as a linear algebra over R with n^2 basis numbers ϵ_{ij} having the multiplication table

$$\epsilon_{ij}\epsilon_{jk} = \epsilon_{ik}, \qquad \epsilon_{ij}\epsilon_{lk} = 0 \qquad j \neq l.$$

In fact, the total matric algebra may be defined as such a linear algebra, and it may be proved that such an algebra can be represented by $n \times n$ arrays of numbers of R with the operations of addition, scalar multiplication and multiplication as we have defined them.

The total matric algebra M may contain a subalgebra of order less than n^2. Thus all linear combinations of the matrices

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with coefficients in the real field form a subalgebra of order 2 of the total matric algebra of order 2^2. This algebra is isomorphic with the complex field under the correspondence

$$a + bi \leftrightarrow aI + bJ = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

This is an instance of the very general theory of matric representation which was glimpsed by Laguerre [5]. Subsequently it was proved by H. Poincaré [8] that every associative algebra of finite order n over a ring R with unit element can be represented as an algebra of $n \times n$ matrices.

3. Endomorphisms. It was remarked in the first section that Frobenius encountered matric algebra by way of bilinear form theory, and his early work demonstrates that matrices may be introduced in this way in a logically satisfactory manner. The approach is not altogether satisfactory psychologically, however, for the composition of bilinear forms, as Frobenius defined it, has something of the appearance of artificiality.

Another approach to the theory of matrices which has recently been gaining favor [9, 10] is somewhat sophisticated but sound both logically and psychologically. Not only is it obvious how the sum and product of two matrices must be defined, but their laws of combination are almost self evident.

A vector space [11] is a commutative group with an operator domain F which is a field. That is, the space V consists of elements $\alpha, \beta, \cdots$, which form a commutative group with respect to addition, and this group admits an operation, called scalar multiplication, such that for all operators (numbers of F) k and l

$$k\cdot(\alpha+\beta)=k\cdot\alpha+k\cdot\beta, \qquad (k+l)\cdot\alpha=k\cdot\alpha+l\cdot\alpha,$$
$$(kl)\cdot\alpha=k\cdot(l\cdot\alpha), \qquad 1\cdot\alpha=\alpha.$$

An endomorphism $\mathfrak{a}$ of a vector space V is a correspondence

$$\alpha\rightarrow\alpha',\ \beta\rightarrow\beta',\cdots$$

of the elements of V such that

$$\alpha+\beta\rightarrow\alpha'+\beta', \qquad k\cdot\alpha\rightarrow k\cdot\alpha', \qquad k \text{ in } F.$$

While every vector α corresponds to a unique vector α', it is not necessary that every vector α' be the correspondent of a vector α. Thus the correspondence

$$\alpha\rightarrow 0,\ \beta\rightarrow 0,\cdots$$

is an endomorphism, called the zero endomorphism.

The endomorphism $\mathfrak{a}$ may be thought of as being brought about by an operator, and from this point of view we write

$$\mathfrak{a}\alpha=\alpha',\ \mathfrak{a}\beta=\beta',\cdots.$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are two such operators, the correspondence

$$\alpha\rightarrow\mathfrak{a}\alpha+\mathfrak{b}\alpha$$

may readily be shown to be an endomorphism, which we define to be the sum of the endomorphisms $\mathfrak{a}$ and $\mathfrak{b}$. Thus by definition

$$(\mathfrak{a}+\mathfrak{b})\alpha=\mathfrak{a}\alpha+\mathfrak{b}\alpha.$$

The set of all endomorphisms of V form a commutative group with respect to addition, the zero endomorphism being the identity.

If $\mathfrak{a}$ and $\mathfrak{b}$ are two endomorphisms, the correspondence

$$\alpha\rightarrow\mathfrak{a}(\mathfrak{b}\alpha)$$

may be shown to be an endomorphism. This is called the product of the two endomorphisms $\mathfrak{a}$ and $\mathfrak{b}$, so that by definition

$$(\mathfrak{a}\mathfrak{b})\alpha=\mathfrak{a}(\mathfrak{b}\alpha).$$

The identity endomorphism

$$\alpha \to \alpha, \ \beta \to \beta, \cdots$$

is the identity element of multiplication.

It can be proved very easily and naturally that multiplication of endomorphisms is associative, and distributive with respect to the addition of endomorphisms. Thus, let α be any vector of V, and $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ any three endomorphisms. Then by the definition of product

$$[\mathfrak{a}(\mathfrak{b}\mathfrak{c})]\alpha = \mathfrak{a}[(\mathfrak{b}\mathfrak{c})\alpha] = \mathfrak{a}[\mathfrak{b}(\mathfrak{c}\alpha)],$$

$$[(\mathfrak{a}\mathfrak{b})\mathfrak{c}]\alpha = (\mathfrak{a}\mathfrak{b})(\mathfrak{c}\alpha) = \mathfrak{a}[\mathfrak{b}(\mathfrak{c}\alpha)].$$

Since the endomorphisms $\mathfrak{a}(\mathfrak{b}\mathfrak{c})$ and $(\mathfrak{a}\mathfrak{b})\mathfrak{c}$ have the same effect on every vector α of V, they are equal.

It is now clear that the set of all endomorphisms of V is a ring with unit element. *A matrix over a field F may be defined as an endomorphism of a vector space V over F or, alternatively, as the operator which produces this endomorphism.*

For matrices of finite order, the equivalence of this definition with that of Section 2 is not hard to establish. Every vector space V of order n over the field F can be represented by n-tuples

$$\alpha = (a_1, a_2, \cdots, a_n), \qquad \beta = (b_1, b_2, \cdots, b_n), \cdots$$

of numbers of F. The relation $\alpha = \beta$ means that

$$a_i = b_i \qquad i = 1, 2, \cdots, n.$$

Addition and scalar multiplication are represented as follows:

$$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n),$$

$$k \cdot \alpha = (ka_1, ka_2, \cdots, ka_n) \qquad k \text{ in } F.$$

The vectors

$$\epsilon_1 = (1,0, \cdots, 0), \ \epsilon_2 = (0, 1, \cdots, 0), \cdots, \ \epsilon_n = (0, 0, \cdots, 1)$$

form a basis for V so that every vector α can be uniquely written

$$\alpha = a_1\epsilon_1 + a_2\epsilon_2 + \cdots + a_n\epsilon_n.$$

An endomorphism $\mathfrak{a}$ of V is uniquely determined by its effect upon the basis vectors. Let

$$\mathfrak{a}\epsilon_i = \sum_{j=1}^{n} a_{ij}\epsilon_j \qquad j = 1, 2, \cdots, n,$$

and consider the correspondence

$$\mathfrak{a} \to (a_{ij}).$$

If $\mathfrak{b}$ is another endomorphism, and if $\mathfrak{b} \to (b_{ij})$, then $\mathfrak{a} = \mathfrak{b}$ if and only if

$$a_{ij} = b_{ij}, \qquad i, j = 1, 2, \cdots, n.$$

If a_{ij} are any n^2 elements of F, (a_{ij}) determines an endomorphism, so that the correspondence $\mathfrak{a} \leftrightarrow (a_{ij})$ is biunique.

It is almost immediate that

$$\mathfrak{a} + \mathfrak{b} \leftrightarrow (a_{rs} + b_{rs}), \qquad \mathfrak{a} \times \mathfrak{b} \leftrightarrow (\sum b_{ri}a_{is})$$

so that, if we define

$$(a_{rs}) + (b_{rs}) = (a_{rs} + b_{rs}), \qquad (a_{rs}) \times (b_{rs}) = (\sum a_{ri}b_{is}),$$

the correspondence $\mathfrak{a} \leftrightarrow (a_{rs})$ is an isomorphism.

It has been remarked that the laws of combination of endomorphisms, such as the associative and distributive laws, are very easily and naturally provable. Since matrices (in the sense of Section 2) are isomorphic with endomorphisms, these same laws of combination must therefore hold for matrices.

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THE HEAVISIDE OPERATIONAL CALCULUS

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1. Introduction. Operational methods for dealing with certain types of problems form a part of most college courses on formal solutions of differential equations. These methods go far back in the history of analysis; certain germs of ideas on this subject are found in Leibniz; and they were particularly interesting to the mathematicians of the middle of the nineteenth century.† Toward the close of that century Oliver Heaviside, in the course of his engineering investigations, extended these methods in a bold and independent manner. He did not always stop to justify his conclusions from the point of view of mathematical rigor; and indeed for him, since the answer to a practical problem can usually be checked by other means, such a justification was doubtless a second-

* This paper is the substance of an address delivered, by invitation, before the Philadelphia Section of the Association on November 28, 1942.

† For the history of the subject see Davis and Gardner (works cited in the bibliography at the end of the paper).

ary matter. The result is that Heaviside handed to mathematicians a problem, *viz.*, the theoretical systematization of his methods, so that they become not only an engine of empirical discovery, but an instrument of mathematical proof.

This problem has been essentially solved for at least ten years. In fact a distinguished colleague has stated, in conversation, that the Heaviside operational calculus is no longer of any mathematical interest; indeed further, that the best engineers have, even in their practice, completely replaced the Heaviside calculus with the theory of Laplace or Fourier transforms. Be that as it may, I know from the experience of the last six months that there are engineers who find the theory of integral transforms involves difficulties which they would prefer to avoid. Moreover there are serious engineering problems, *viz.*, those having to do with discrete mechanisms or networks, for which the difficulties of the integral transforms appear to be irrelevant. There is, consequently, some interest in a theory of the Heaviside calculus of a more elementary character.

Accordingly, the calculus is here discussed from a point of view which is essentially algebraic rather than analytical. This, of course, implies a restriction on the scope of the treatment, because it is limited to the rational aspects such as arise from ordinary linear differential equations with constant coefficients. For the more general cases of partial differential equations, fractional operators, *etc.*, the theory of integral transforms is doubtless unavoidable. However the treatment is sufficient for the engineering applications just alluded to. Moreover the theory treats with ease those cases of discrete systems which are commonly called "impulsive,"* such as electrical networks with a pure capacitance in some member; in these cases the theory may conceivably overcome difficulties which are not met by the theory of integral transforms. Presumably the two methods can be combined; but that question is not gone into here.

The discussion is, at least from one point of view, more elementary than that requiring infinite integrals. On the other hand, since the paper is addressed to mathematicians, I have not hesitated to use some of the simpler concepts of modern algebra. These give the work a certain abstractness and generality. Thus, although this subject has a bearing on the elementary teaching of differential equations, an exposition intelligible to college juniors is not attempted. However, the presentation is doubtless intelligible for college teachers; and, except for the concepts mentioned and the omission of certain proofs, it is intended to be self contained.

2. Generalities concerning linear operators. We consider a certain set $\mathfrak{L}$ of operators over a space $\mathfrak{A}$. *I.e.*, if L is in $\mathfrak{L}$ and A is in $\mathfrak{A}$, then $L[A]$, the result of applying the operator L to A, is a unique element of $\mathfrak{A}$. In the sequel the argument of an operator will be enclosed in square brackets; when this operand is itself a function, the argument of this function will be indicated by a subscript, thus: $L[f(t)]_t$. When the operator itself is designated by a complex expression, that expression will be enclosed in braces ("$\{$," "$\}$"); thus $\{L_1+L_2\}[A]$ is the

* The term "impulsive" in this connection is taken from Bush (see the bibliography).

result of applying the operator L_1+L_2 to A.* However braces or brackets will be omitted in certain cases where their explicit appearance seems superfluous.

Concerning the space $\mathfrak{A}$ we shall suppose that it is a linear vector space; *i.e.*, that it is an Abelian group with respect to addition, and that it admits multiplication by scalars (complex numbers), satisfying the usual postulates.

Under these circumstances the usual combinations of operators may be defined thus (the derived operators being defined in terms of their effects on an unspecified operand A):

$$\{L_1 + L_2\}[A] = L_1[A] + L_2[A]. \tag{1}$$

$$\{L_1L_2\}[A] = L_1[L_2[A]]. \tag{2}$$

We can also define multiplication of operators by scalars; but this is unnecessary if we suppose that our set $\mathfrak{L}$ contains, for every scalar c, the operation of multiplying by c. This operation will be denoted by "$\{c\}$," so that we have

$$\{c\}[A] = cA. \tag{3}$$

Ordinarily the braces will be omitted, so that the same symbol is to be interpreted as an operation or a scalar according to the context.

A *linear* operator is one which satisfies the following:

$$L[A_1 + A_2] = L[A_1] + L[A_2], \tag{4}$$

and for any scalar c

$$L[cA] = cL[A]. \tag{5}$$

THEOREM 1. *If $\mathfrak{L}$ is a set of linear operators which is closed under addition and multiplication, then $\mathfrak{L}$ is a ring.*

THEOREM 2. *If $\mathfrak{L}_0$ is a set of permutable linear operators, and $\mathfrak{L}$ is the ring generated by $\mathfrak{L}_0$,† then $\mathfrak{L}$ is a commutative ring.*

* The notation is a modification of that used by A. Church in his earlier work on mathematical logic (see "Sets of postulates for the foundation of logic," Annals of Mathematics, vol. 33, 1932, pp. 346–366). The notation is also useful in connection with ordinary functions symbolized by complex symbols, the argument in this case being enclosed in parentheses and the function in braces. Thus $\{L[f(t)]_t\}(t+h)$ means the result of substituting $t+h$ for t in $L[f(t)]_t$. This is, in general, distinct from $L[f(t+h)]_t$, which is the result of operating on $f(t+h)$ regarded as a function of t. In fact if

$$Lf(t) = \frac{f(t) - f(0)}{t} \quad \text{for} \quad t \neq 0$$

$$= f'(0) \quad \text{for} \quad t = 0$$

and $f(t)=t^2$, then $\{L[f(t)]_t\}(t+h)=t+h$, $L[f(t+h)]_t=t+2h$. A considerable amount of work on symbolic operators would probably be made clearer by the systematic adoption of some such notation. The other notation of Church, whereby, if M is any expression containing t, $\lambda t.M$ is that function which assigns to any given t the value M, is also convenient; but it is not used here on account of its too great divergence from ordinary notations. For the present paper the notation explained in the text suffices.

† *I.e.*, the smallest ring which includes $\mathfrak{L}_0$.

The proofs of these theorems will not be given here. We may note, however, that in Theorem 1 the only law which requires that the operators be linear is the distributive law.

The interesting case of Theorem 2 is where $\mathfrak{A}$ is the set of infinitely differentiable functions of a real variable and $\mathfrak{L}_0$ consists of D (differentiation) together with $\{c\}$ for every scalar c. In this case $\mathfrak{L}$ is the ring of polynomials in D with scalar coefficients. The assertion of Theorem 2, then, is that these polynomials obey the rules of manipulation of ordinary algebra. This justifies the applications of symbolic methods made à la Boole in elementary courses in differential equations; likewise it has been shown that most of the algebraic part of the Heaviside theory, in particular his "expansion theorem," can be justified by this means.*

The essential content of the Heaviside theory, however, is that $\mathfrak{A}$ and $\mathfrak{L}$ can be so modified that the resulting set of operators forms a field. This has somewhat the same advantages as the introduction of negative numbers into algebra. I shall not elaborate on these advantages here, but shall proceed at once to derive sufficient conditions that a set of operators may constitute a field.

THEOREM 3. *If L is a linear operator such that for every B in $\mathfrak{A}$, the equation*

$$L[X] = B$$

is satisfied by one and only one X in $\mathfrak{A}$; then there exists a linear operator L^{-1} called the inverse of L such that the equations

$$B = L[A]$$
$$A = L^{-1}[B]$$

are equivalent.

Proof. Let $L^{-1}[B]$ be the unique solution of the equation

$$L[X] = B.$$

Then $L^{-1}[B]$ is defined for every B in $\mathfrak{A}$. To prove L^{-1} is linear, let $B=B_1+B_2$, $A_1=L^{-1}[B_1]$, $A_2=L^{-1}[B_2]$. Then

$$B_1 = L[A_1], \qquad B_2 = L[A_2],$$
$$B = L[A_1] + L[A_2] = L[A_1 + A_2],$$
$$A_1 + A_2 = L^{-1}[B_1] + L^{-1}[B_2] = L^{-1}[B_1 + B_2].$$

Again, let $B=cB_1$, $A_1=L^{-1}[B_1]$, $A=cA_1$. Then, since L is linear,

$$L[A] = cL[A_1] = cB_1 = B.$$

Therefore

$$A = L^{-1}[cB_1] = cL^{-1}[B_1], \text{ q.e.d.}$$

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* For references see footnote[214] in Doetsch.

COROLLARY. *The inverse has the following properties*:

$$LL^{-1} = L^{-1}L = I, \tag{6}$$

where I is the identity operator defined by

$$I[A] = A; \tag{7}$$

also

$$(L^{-1})^{-1} = L. \tag{8}$$

Proof. The first of these is clear. The proof of the theorem shows that L^{-1} is linear; further it has a unique inverse, since the equation $L^{-1}[B] = A$ is equivalent to $L[A] = B$. The rest follows by the symmetry of conditions on L and L^{-1}.

THEOREM 4. *If L_1 and L_2 are operators with unique inverses, then L_1L_2 has a unique inverse, viz.,*

$$(L_1L_2)^{-1} = L_2^{-1}L_1^{-1}; \tag{10}$$

furthermore, if L_1, L_2 are permutable, then any two of the four operators L_1, L_2, L_1^{-1}, L_2^{-1} are permutable.

Proof. By the usual arguments we can show that if L_1 and L_2 have unique inverses, then L_1L_2 has a unique inverse, *viz.*, $L_2^{-1}L_1^{-1}$. Hence, if L_1 and L_2 are permutable,

$$L_1^{-1}L_2^{-1} = (L_2L_1)^{-1} = (L_1L_2)^{-1} = L_2^{-1}L_1^{-1}.$$

To prove the theorem it remains to show that

$$L_1L_2^{-1} = L_2^{-1}L_1, \qquad L_2L_1^{-1} = L_1^{-1}L_2.$$

From symmetry it is sufficient to prove the first of these. This we do as follows:

$$L_1L_2^{-1} = L_2^{-1}L_2L_1L_2^{-1} = L_2^{-1}L_1L_2L_2^{-1} = L_2^{-1}L_1.$$

THEOREM 5. *If p is a linear operator such that for every scalar c the operator $\{p-c\}$ has a unique inverse; then the set $\mathfrak{L}_0$, consisting of $\{p-c\}$ and $\{p-c\}^{-1}$ for all scalars c, generates a field.**

Proof. It is clear that for any scalars c_1 and c_2 the operators $\{p-c_1\}$ and $\{p-c_2\}$ are permutable. Hence, by theorems 2 and 4, $\mathfrak{L}_0$ generates a commutative ring $\mathfrak{L}$. This ring evidently contains a unit element, *viz.*, the identity operator pp^{-1}. It remains to show that every operator in $\mathfrak{L}$ has also an inverse in $\mathfrak{L}$.

If $\phi(p)$ is any polynomial (over the field of complex numbers) $\phi(p)$ can be factored into linear factors thus

$$\phi(p) = c_0(p - c_1)(p - c_2) \cdots (\phi - c_n).$$

Hence $\phi(p)$ has an inverse by Theorem 4. We show by induction that every operator in $\mathfrak{L}$ is of the form $\{\phi(p)\}\{\psi(p)\}^{-1}$ and has the inverse $\{\psi(p)\}\{\phi(p)\}^{-1}$.

* *I.e.*, the ring generated by $\mathfrak{L}_0$ is a field.

3. The system $\mathfrak{H}$. We proceed to specialize the space $\mathfrak{A}$ and the operator p.

The first specialization suggested is that where $\mathfrak{A}$ is the class of functions $f(t)$ of class C^∞ for all real t, and p is D (*i.e.*, differentiation). In this case, however, p does not satisfy the conditions on $\mathfrak{L}$ in Theorem 3 unless we restrict $\mathfrak{A}$ so that, for example, $f(t)$ is in $\mathfrak{A}$ only when $f(0)=0$.* But when this is done p is, in general, no longer an operator on $\mathfrak{A}$ to $\mathfrak{A}$. In fact we do not get an interpretation of the theorems of §2 in the way suggested unless we restrict $\mathfrak{A}$ so that every $f(t)$ in $\mathfrak{A}$ vanishes at 0 together with all its derivatives.†

The specialization adopted here is the following: Let $\mathfrak{F}$ be the class of functions $f(t)$ of class C^∞. Let $\eta_0, \eta_1, \cdots$ be an infinite set of indeterminates like the units of a hypercomplex algebra. Let $\mathfrak{A}$ be the set of all finite sums of the form

$$A = a(t)\eta_0 + a_1\eta_1 + \cdots + a_m\eta_m, \tag{11}$$

where $a(t)$ is in $\mathfrak{F}$ and $a_1, a_2, \cdots, a_m$ are scalars. The number m may vary with A.‡ Finally let p be defined by the requirement that, for A as in (11),

$$p[A] = a'(t)\eta_0 + a_0\eta_1 + \cdots + a_m\eta_{m+1}, \tag{12}$$

where $a_0=a(0)$.

In this case it is clear that $\mathfrak{A}$ is a linear space and p an operator on $\mathfrak{A}$ to $\mathfrak{A}$. That the hypotheses of Theorem 5 are satisfied will be shown as follows:

* The equation $D[x(t)]_t=a(t)$ always has the solution

$$x(t) = \int_0^t a(\tau)d\tau;$$

but it also has other solutions unless these are excluded from $\mathfrak{A}$ by some such restriction as that indicated.

† Hazeltine (see the bibliography) proposes what amounts to considering the set of functions having these properties at $t=-\infty$. However, for physical applications we are interested in functions with discontinuities at $t=0$, and for these the formalism considered below has advantages.

‡ Since it is not convenient to impose the condition $a_m\neq 0$, m is not even uniquely determined by A.

The formulation in the text is, of course, equivalent to the following: Let $\mathfrak{A}$ be the class of all infinite sequences A of the form

$$A = [a(t), a_1, a_2, \cdots],$$

where $a(t)$ is in $\mathfrak{F}$, the a_i are constants, and not more than a finite number of the a_i are different from 0. Sums and products by scalars are defined thus:

$$[a(t), a_1, a_2, \cdots] + [b(t), b_1, b_2, \cdots] = [a(t) + b(t), a_1 + b_1, a_2 + b_2, \cdots]$$
$$c[a(t), a_1, a_2, \cdots] = [ca(t), ca_1, ca_2, \cdots].$$

Then the indeterminates η_i are defined thus

$$\eta_0 = [1, 0, 0, \cdots]$$
$$\eta_1 = [0, 1, 0, 0, \cdots]$$
$$\eta_2 = [0, 0, 1, 0, 0 \cdots], \quad \text{etc.},$$

and it is a theorem that every A and $\mathfrak{A}$ has the form (11) if we add the special definition

$$a(t)\eta_0 = [a(t), 0, 0, \cdots].$$

(A more general definition of $b(t)A$, for $b(t)$ in $\mathfrak{F}$ and A in $\mathfrak{H}$, is not needed in this paper.)

THEOREM 6. *If A and p are defined as above, then for every scalar c the equation*

$$\{p - c\}[X] = A \tag{13}$$

has a unique solution.

Proof. Let A be given as above and suppose that

$$X = x(t)\eta_0 + x_1\eta_1 + \cdots + x_n\eta_n.$$

Then

$$\begin{aligned} p[X] &= x'(t)\eta_0 + x(0)\eta_1 + x_1\eta_2 + \cdots + x_n\eta_{n+1}, \\ \{p - c\}[X] &= (x'(t) - cx(t))\eta_0 + (x(0) - cx_1)\eta_1 + (x_1 - cx_2)\eta_2 + \cdots \\ &\quad + (x_{n-1} - cx_n)\eta_n + x_n\eta_{n+1}. \end{aligned}$$

We may suppose without loss of generality that $n > m$ (since we have not assumed $x_n \neq 0$). Then in order to satisfy (13) it is necessary and sufficient that

$$\begin{aligned} x'(t) - cx(t) &= a(t), \\ x(0) - cx_1 &= a_1, \\ x_1 - cx_2 &= a_2, \\ &\cdots\cdots\cdots \\ x_{m-1} - cx_m &= a_m, \\ x_m - cx_{m+1} &= 0, \\ &\cdots\cdots\cdots \\ x_{n-1} - cx_n &= 0, \\ x_n &= 0. \end{aligned}$$

These equations have a unique solution. Working in inverse order we have

$$\begin{aligned} x_n &= 0, \\ x_{n-1} &= 0, \\ &\cdots\cdots \\ x_m &= 0, \\ x_{m-1} &= a_m, \\ x_{m-2} &= a_{m-1} + ca_m, \\ &\cdots\cdots\cdots \\ x_{m-k} &= a_{m-k+1} + ca_{m-k} + c^2a_{m-k-1} + \cdots + c^{k-1}a_m, \\ x(0) &= a_1 + ca_2 + \cdots + c^{m-1}a_m, \\ x(t) &= x(0)e^{ct} + \int_0^t e^{c(t-\tau)}a(\tau)d\tau. \end{aligned} \tag{14}$$

This is evidently a solution, and the only one. For $c=0$ we have

$$\frac{1}{p}[A] = \left(a_1 + \int_0^t a(\tau)d\tau\right)\eta_0 + a_2\eta_1 + \cdots + a_m\eta_{m-1}.$$

This completes the proof of Theorem 6. It then follows, by the argument of §2, that the set $\mathfrak{H}$ of operators of the form $\{\phi(p)\}\{\psi(p)\}^{-1}$ constitutes a field.

In interpretation η_0 may be thought of as the Heaviside unit step function, η_1 as the unit impulse, and $\eta_2, \eta_3, \cdots$ as impulses of higher order. However other interpretations are possible; in particular it is not necessary that $a(t)\eta_0$ be thought of as a function which is 0 for all $t<0$.

Let us now consider two theorems concerning the system $\mathfrak{H}$.

The first of these has to do with the concept of *rank*, which is defined as follows: Let A be given by (11) and let

$$a_0 = a(0),$$
$$a_{-k} = a^{(k)}(0), \qquad k = 1, 2, 3, \cdots;$$

then the rank of A is the greatest r such that $a_r \neq 0$.* We then have

THEOREM 7. *If A is of rank r, then for any scalar c, $\{p-c\}[A]$ is of rank $r+1$.*

The final theorem concerns a connection between the system $\mathfrak{H}$ and ordinary differential operators. Let us say that an element A of $\mathfrak{A}$ is *congruent* to a function $f(t)$ of $\mathfrak{F}$, in symbols

$$A \cong f(t), \tag{15}$$

if and only if

$$a(t) = f(t),$$

where A is given by (11). Then we have the following:

THEOREM 8. *If $\phi(p)$ is a polynomial in p, and*

$$A \cong f(t);$$

then

$$\{\phi(p)\}[A] \cong \phi(D)\}[f(t)]_+. \tag{16}$$

The proof of this is merely a matter of formal manipulation.

4. Application to differential equations. In the following bold face letters will be used to denote matrices of n rows and either n columns or one column according to the context. The presentation of the applications will be informal, and proofs will generally be omitted.

* Note that the rank can be any integer positive, negative, or zero. If no r satisfying the conditions exists the rank is not defined.

Let us consider a system of differential equations of the form

$$\{\mathbf{Z}(D)\}[\mathbf{x}(t)]_t = \mathbf{g}(t), \tag{17}$$

where $\mathbf{Z}(D)$ is a matrix polynomial in D of degree q with constant coefficients. If $\mathbf{x}(t)$ is any solution of (17) and if

$$\mathbf{X} = \mathbf{x}(t)\eta_0 + \mathbf{x}_1\eta_1 + \cdots + \mathbf{x}_r\eta_r; \tag{18}$$

then by Theorem 8

$$\{\mathbf{Z}(p)\}[\mathbf{X}] \cong \mathbf{g}(t),$$

and hence $\mathbf{X}$ is a solution of the equation

$$\{\mathbf{Z}(p)\}[\mathbf{X}] = \mathbf{G}, \tag{19}$$

where

$$\mathbf{G} = \mathbf{g}(t)\eta_0 + \mathbf{g}_1\eta_1 + \cdots + \mathbf{g}_s\eta_s. \tag{20}$$

$\mathbf{g}_1, \mathbf{g}_2, \cdots$ have values determined by $\mathbf{x}$.* Conversely suppose $\mathbf{g}_1, \mathbf{g}_2, \cdots$ are given arbitrary values and that $\mathbf{X}$ is a solution of (19); then $\mathbf{x}(t)$ is a solution of (17). However the equation (19) always has a unique solution provided only that the determinant of $\mathbf{Z}(p)$ is not zero; this solution is given by

$$\mathbf{X} = \{(\mathbf{Z}(p))^{-1}\}[\mathbf{G}],$$

where $(\mathbf{Z}(p))^{-1}$ is the inverse matrix of $\mathbf{Z}(p)$. It should be noted that this argument gives a rigorous proof of the existence of a solution of (17) under these circumstances.

Let us suppose now that

$$\mathbf{Z}(D) = \mathbf{z}_0 + \mathbf{z}_1 D + \mathbf{z}_2 D^2 + \cdots + \mathbf{z}_q D^q \tag{21}$$

and that the determinant of $\mathbf{z}_q$ does not vanish. Such systems will be called *non-impulsive.*† In that case every element of the matrix $(\mathbf{Z}(p))^{-1}$ will be a rational function of p whose denominator is of degree nq and whose numerator is at most of degree $(n-1)q$. It follows, by reference to the argument of Theorem 7, that $(\mathbf{Z}(p))^{-1}$ decreases the rank of its argument by at least q units. Hence if $\mathbf{g}_1 = \mathbf{g}_2 = \cdots = 0$, $\mathbf{G}$ is at most of rank 0 and $\mathbf{X}$ is at most of rank $-q$. Therefore, the $\mathbf{x}(t)$ determined in this way is precisely the solution of (17) such that

$$\mathbf{x}(0) = \mathbf{x}'(0) = \cdots = \mathbf{x}^{(q-1)}(0) = 0. \tag{22}$$

It may be shown that the solution such that

$$\mathbf{x}(0) = \mathbf{c}_0, \ \mathbf{x}'(0) = \mathbf{c}_1, \cdots, \mathbf{x}^{(q-1)}(0) = \mathbf{c}_{q-1} \tag{23}$$

is obtained by putting

* The explicit formulas are (38), given in the appendix.

† In this case the equations may be solved for the highest derivatives.

$$
\begin{aligned}
\mathbf{g}_1 &= \mathbf{z}_1\mathbf{c}_0 + \mathbf{z}_2\mathbf{c}_1 + \cdots + \mathbf{z}_q\mathbf{c}_{q-1}, \\
\mathbf{g}_2 &= \mathbf{z}_2\mathbf{c}_0 + \mathbf{z}_3\mathbf{c}_1 + \cdots + \mathbf{z}_q\mathbf{c}_{q-2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathbf{g}_{q-1} &= \mathbf{z}_{q-1}\mathbf{c}_0 + \mathbf{z}_q\mathbf{c}_1, \\
\mathbf{g}_q &= \mathbf{z}_q\mathbf{c}_0.
\end{aligned} \tag{24}
$$

The effect of adding additional impulsive terms to $\mathbf{G}$ is merely to add impulsive terms to $\mathbf{X}$.*

In the impulsive case, where the determinant of $\mathbf{z}_q$ is zero, the situation is more complicated. Since certain of the derivatives can be eliminated, the initial conditions (23) cannot be given arbitrarily. In extreme cases $\{\mathbf{Z}(p)\}^{-1}$ may actually increase the rank of its argument, so that the output ($\mathbf{X}$) may be impulsive (rank >0) even when the input ($\mathbf{G}$) is not. Such a phenomenon occurs in the case of a condenser, where

$$
\begin{aligned}
\dot{q}(t) - i(t) &= 0 \\
q(t) &= ce(t).\dagger
\end{aligned} \tag{25}
$$

Nevertheless in all these cases the argument of the second preceding paragraph shows that if (17) has a solution at all that solution is obtainable from (19). Moreover the interpretation of the η_i, given in §3, suggests that the solution for $\mathbf{g}_1=\mathbf{g}_2=\cdots=0$ gives the output when the input is suddenly applied to a system initially undisturbed (there being in that case certain discontinuities at $t=0$).‡

The practical determination of the solution of (19) can be accomplished by the use of (12) and (14) if the determinant of $\mathbf{Z}(p)$ can be factored into linear factors.§ This process may be shortened by developing the elements of the matrix $(\mathbf{Z}(p))^{-1}$ in partial fractions. The "Heaviside expansion theorem" is

---

* For a somewhat fuller statement see the appendix below.

† These equations form an example of the preceding theory for the following specializations of $\mathbf{x}(t)$, $\mathbf{Z}(D)$, $\mathbf{g}(H)$:

$$
\mathbf{x}(t) = \begin{bmatrix} q(t) \\ i(t) \end{bmatrix}
$$

$$
\mathbf{Z}(D) = \begin{bmatrix} D & -1 \\ 1 & 0 \end{bmatrix}
$$

$$
\mathbf{g}(t) = \begin{bmatrix} 0 \\ ce(t) \end{bmatrix}.
$$

Of course the example is trivial in that the second equation is already solved for $q(t)$ and the first gives by inspection

$$
i(t) = \dot{q}(t) = c\dot{e}(t)
$$

‡ For further discussion of the impulsive case see the appendix.

§ There are, of course, other methods, such as expansions in series, which are not treated here.

merely a formula expressing the result of the process in a rather special case.*

5. The superposition theorem. Up to the present we have made no assumption in regard to the input. In the usual treatment of the Heaviside calculus it is traditional to solve the problem first for a special case, obtaining the so called "indicial admittance," and then to express the general solution by means of an integral. Although it is not clear that this detour is always advantageous, yet there is sufficient interest in it to justify considering the form which this theorem takes in the most general impulsive case. On this as a basis one can then proceed to establish contact with the Carson integral equation and so to infinite integral forms of the Heaviside theory.

In the present matrix notation the *indicial admittance* is to be defined as the matrix $\mathbf{A}$ such that

$$\{\mathbf{Z}(p)\}[\mathbf{A}] = \mathbf{i}\eta_0$$

where $\mathbf{i}$ is the unit matrix. If this indicial admittance is given by

$$\mathbf{A} = \mathbf{a}(t)\eta_0 + \mathbf{a}_1\eta_1 + \cdots + \mathbf{a}_m\eta_m,$$

then it may be shown that

$$(\mathbf{Z}(p))^{-1} = \mathbf{B}_0(p) + \mathbf{a}_0 + \mathbf{a}_1 p + \cdots + \mathbf{a}_m p^m, \tag{26}$$

where $\mathbf{a}_0 = \mathbf{a}(0)$ and $\mathbf{B}_0(p)$ is an operation which decreases the rank. Again, if $\mathbf{X}$ is given by (19), it may also be shown that

$$\begin{aligned}\mathbf{X} = &\left(\mathbf{a}(t)\mathbf{g}(0) + \int_0^t \mathbf{a}(t-\tau)\mathbf{g}'(\tau)d\tau\right)\eta_0 \\ &+ \mathbf{a}_1 p[\mathbf{g}(t)\eta_0] + \mathbf{a}_2 p^2[\mathbf{g}(t)\eta_0] + \cdots \\ &+ \mathbf{a}_m p^m[\mathbf{g}(t)\eta_0] + p[\mathbf{A}]\mathbf{g}_1 + p^2[\mathbf{A}]\mathbf{g}_2 + \cdots + p^s[\mathbf{A}]\mathbf{g}_s.\end{aligned} \tag{27}$$

This is the principle of superposition for the most general impulsive system. The result is valid if $\mathbf{Z}(p)$ is any matrix rational in p.

* The theory of the general linear homogeneous differential system with constant coefficients, *viz.*,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{a}\mathbf{x}(t),$$

where $\mathbf{a}$ is a constant matrix, can be based on the foregoing discussion for the case $\mathbf{G} = \mathbf{c}\eta_1$. Such a treatment has many advantages over the orthodox methods. In fact the latter involve, for the treatment of the general case, formal difficulties as advanced as those of the theory of elementary divisors. But the most general case, including all existence and uniqueness theorems, can be handled by the techniques mentioned in the text together with the formula

$$\{p - c\}^{-k}[\eta_1] = \frac{t^{k-1}}{(k-1)!}e^{ct}\eta_0,$$

which follows from (14). This supports the contention that the Heaviside calculus can be a tool of theoretical, as well as practical, investigation.

To proceed from here it is advisable to transform the integral by integration by parts and change of the variable from τ to $t-\tau$. Since the impulsive terms will not further concern us, we may express the result in the form of a congruence, thus

$$\mathbf{X} \cong \int_0^t \mathbf{a}'(\tau)\mathbf{g}(t-\tau)d\tau + \sum_{i=0}^{m} \mathbf{a}_i\mathbf{g}^{(i)}(t) + \sum_{j=1}^{s} \mathbf{a}^{(j)}(t)\mathbf{g}_j. \tag{28}$$

Next we take the special case

$$\mathbf{G} = \mathbf{g}_0 e^{\lambda t}\eta_0. \tag{29}$$

Then the above formula becomes

$$\mathbf{X} \cong \left(\int_0^t \mathbf{a}'(\tau)e^{-\lambda\tau}d\tau + \sum_{i=0}^{m} \mathbf{a}_i\lambda^i \right)\mathbf{g}_0 e^{\lambda t}. \tag{30}$$

If now there exists a constant a_∞ and a real λ_0 such that, as $t\to\infty$

$$\mathbf{a}(t) - \mathbf{a}_\infty \in O(e^{\lambda_0 t}) \tag{31}$$

then for the real part of $\lambda > \lambda_0$, $\mathbf{x}(t)$ is asymptotic as $t\to\infty$ to

$$\mathbf{y}(\lambda)\mathbf{g}_0 e^{\lambda t}$$

where

$$\begin{aligned} \mathbf{y}(\lambda) &= \int_0^\infty \mathbf{a}'(\tau)e^{-\lambda\tau}d\tau + \sum_{i=0}^{m} \mathbf{a}_i\lambda^i \\ &= \lambda\int_0^\infty (\mathbf{a}(\tau) - \mathbf{a}_\infty)e^{-\lambda\tau}d\tau + \mathbf{a}_\infty + \sum_{i=1}^{m} \mathbf{a}_i\lambda^i. \end{aligned} \tag{32}$$

This situation will be expressed by the notation

$$\mathbf{X} \sim \mathbf{y}(\lambda)\mathbf{g}_0 e^{\lambda t}, \tag{33}$$

where "$\sim$" is to be read "is asymptotically congruent to."

To get another asymptotic formula we must use a generalization of Theorem 8, which, since it concerns the underlying system $\mathfrak{H}$, will be stated as follows:

THEOREM 9. *If $\phi(p)$, $\psi(p)$ are polynomials in p, A is in $\mathfrak{A}$ and is such that*

$$A \cong g(t),$$

and if further $f(t)$ is such that

$$\{\phi(D)\}[g(t)]_t = \psi(D)[f(t)]_t;$$

then

$$\frac{\phi(p)}{\psi(p)}[A] \cong f(t) + h(t),$$

where

$$\{\psi(D)\}[h(t)]_t = 0.$$

The proof of this theorem is again a matter of formal manipulation. In the case where $g(t)=g_0e^{\lambda t}$, where $\psi(\lambda)\neq 0$, the hypotheses are fulfilled for $f(t)=g_0e^{\lambda t}\phi(\lambda)/\psi(\lambda)$. If furthermore the roots of $\phi(\lambda)$ all have negative real parts, $h(t)\to 0$ as $t\to\infty$ and

$$\frac{\phi(p)}{\psi(p)}\,[A]\sim\frac{\phi(\lambda)}{\psi(\lambda)}\,g_0e^{\lambda t}.$$

The preceding argument holds when $\phi(p)$, A, $g(t)$, $f(t)$ (but not $\psi(p)$) are matrices. It can therefore be applied in the case where $\phi(p)/\psi(p)$ is $\{\mathbf{Z}(p)\}^{-1}$ and A is the $\mathbf{G}$ of (19). The restriction on the roots of $\psi(\lambda)$ then corresponds to the requirement that the system (17) be stable. In that case we have, if $\psi(\lambda)\neq 0$,

$$\mathbf{X}\sim[\mathbf{Z}(\lambda)]^{-1}\mathbf{g}_0e^{\lambda t}. \tag{34}$$

If, now, we compare (33) and (34) we have

$$\lim_{t\to\infty}(\mathbf{y}(\lambda)-[\mathbf{Z}(\lambda)]^{-1})e^{\lambda t}=0;$$

and hence, if the real part of λ is non-negative,

$$\mathbf{y}(\lambda)=[\mathbf{Z}(\lambda)]^{-1}. \tag{35}$$

In view of (32) this is the analogue of the Carson integral equation. If $\mathbf{Z}(p)$ is a polynomial in p then (31) is a consequence of the other by hypotheses made. Since $\mathbf{y}(0)=(\mathbf{Z}(0))^{-1}=\mathbf{a}_\infty$ by (32), the result may be written

$$[\mathbf{Z}(\lambda)]^{-1}-[\mathbf{Z}(0)]^{-1}=\lambda\int_0^\infty(\mathbf{a}(\tau)-\mathbf{a}_\infty)e^{-\lambda\tau}d\tau+\sum_{i=1}^{m}\mathbf{a}_i\lambda^i. \tag{36}$$

APPENDIX*

Note on the impulsive case. Inasmuch as Bromwich was unable to solve the general impulsive case, it may be worth while to state some additional facts concerning it. In this statement it is assumed that $\mathbf{Z}(D)$ is a matrix polynomial, as given by (21). Furthermore it will be assumed, without further mention, that we are dealing with solutions of (19) with preassigned $\mathbf{g}(t)$ and that a "solution" is such as to satisfy (17).

First we shall review some of the facts considered in §4. If we define

$$\begin{aligned}&\mathbf{x}_0=\mathbf{x}(0)\\ &\mathbf{x}_{-k}=\mathbf{x}^{(k)}(0),\qquad k=1,2,\cdots,q-1,\\ &\mathbf{z}_k=0\quad\text{for}\quad k<0;\end{aligned} \tag{37}$$

then by direct calculation from (19) we obtain

$$\mathbf{g}_j=\sum_{k=j-r}^{q}\mathbf{z}_k\mathbf{x}_{j-k}. \tag{38}$$

* Not in the address as delivered.

Let us call the right side of (38) $\mathbf{g}_j(\mathbf{x})$, where $\mathbf{x}$ may be thought of as the three dimensional matrix whose plane sections are the $\mathbf{x}_i$. Then a necessary condition that

$$\mathbf{x}_i = \mathbf{c}_i, \qquad i = 1 - q,\ 2 - q, \cdots, \tag{39}$$

is that

$$\mathbf{g}_j(\mathbf{x}) = \mathbf{g}_j(\mathbf{c}). \tag{40}$$

In the non-impulsive case this is also sufficient, because we can show by induction, starting with the largest j, that (40) has the unique solution (39). Since the solution of (1) is unique this sufficiency holds also in the general case, provided only that a solution of (19) satisfying (39) exists. It remains to find what condition must be imposed on the $\mathbf{c}_i$ in order that this should be so.

To investigate this we can suppose that $q=1$, since the general case can be reduced to this one by increasing n. Then by eliminating certain components of $p\mathbf{X}$ it can be shown that (19) is equivalent to a system consisting of ν equations ($0 \leqq \nu < n$) forming a non impulsive sub-system in ν unknowns (which we shall take to be the first ν), together with $n-\nu$ algebraic equations determining the remaining unknowns uniquely in terms of the first ν and the inputs. It can be shown further that a necessary and sufficient condition that a solution of (19) satisfying (39) exist is that the last $n-\nu$ components of $\mathbf{c}_0$ have the values determined by putting $t=0$ in the last $n-\nu$ equations; the $\mathbf{c}_i$ for $i>0$ and the first ν components of $\mathbf{c}_0$ being arbitrary.

It should be noted that the solution for all $\mathbf{g}_j=0$ ($j>0$) does not in general correspond to the case where all the arbitrary components of the $\mathbf{c}_i$ are zero. The explanation of this is that the above elimination may require operating on the right sides of the equations with operators containing p, and hence the new inputs may have impulsive terms when there were none in the originals and vice versa.

The following example illustrates the phenomenon: Consider the system

$$\begin{aligned} \dot{x}_1(t) - \dot{x}_2(t) &= g_1(t), \\ x_2(t) - \dot{x}_3(t) &= g_2(t). \\ x_3(t) &= g_3(t). \end{aligned}$$

The corresponding Heaviside system

$$\begin{aligned} pX_1 - pX_2 &= G_1, \\ X_2 - pX_3 &= G_2, \\ X_3 &= G_3. \end{aligned}$$

has the solution

$$X_1 = \frac{1}{p}G_1 + G_2 + pG_3,$$

$$X_2 = G_2 + pG_3,$$
$$X_3 = G_3.$$

If $G_i = g_i(t)\eta_0 + g_{i1}\eta_1$, then

$$X_1 = \left(\int_0^t g_1(\tau)d\tau + g_2(t) + g_3(t) + g_{11}\right)\eta_0 + (g_3(0) + g_{21})\eta_1 + g_{31}\eta_2,$$
$$X_2 = (g_2(t) + g_3(t))\eta_0 + (g_{21} + g_3(0))\eta_1 + g_{31}\eta_2,$$
$$X_3 = g_3(t)\eta_0 + g_{31}\eta_1.$$

If now $g_{11} = g_{21} = g_{31} = 0$, X_1 and X_2 have the impulsive term $g_3(0)\eta_1$; if we take $g_{11} = g_2(0) - g_3(0)$, $g_{21} = -g_3(0)$, $g_{31} = 0$, then X is nonimpulsive and $x_1(0) = 0$. Evidently $x_2(0) = g_2(0) + g_3(0)$ in any case.

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The following contains a list of items consulted in connection with the above paper. It is not intended to be exhaustive. Several important papers were not available, on account of war conditions, during the writing of this article (see the bibliographies in works by von Karman and Pipes). A few works not bearing directly on the Heaviside calculus are cited in the footnotes, but are not listed here.

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DISCUSSIONS AND NOTES

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ANALOGS

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As has been the case with many discoveries, Fourier series originated, indirectly, from the study of another topic. Viewed directly from a mathematical standpoint, one would expect series of the form,

$$f(x) = \sum_{n=0}^{\infty} a_n(\sin x)^n, \tag{1}$$

$$f(x) = \sum_{n=0}^{\infty} b_n(\cos x)^n, \tag{2}$$

to find their way into text books before the Fourier series since they are the analogs, in trigonometric form, of the Maclaurin expansion. Furthermore, the determination of the coefficients is not difficult, although somewhat tedious.

The coefficients a_n in (1) depend upon the successive derivatives of $(\sin x)^n$, $m > n-1$. These derivative may take any of the following forms:

$$D^m(\sin x)^n = n(\sin x)^{n-m}\phi(\cos x), \qquad m < n,$$

$$D^m(\sin x)^n = n\phi(\cos x), \qquad (m = n + 2p,\ p = 0, 1, 2, \cdots),$$

$$D^m(\sin x)^n = n \sin x\phi(\cos x), \qquad (m = n + 2p + 1,\ p = 0, 1, 2, \cdots),$$

where $\phi(\cos x)$ is a polynomial in $\cos x$ of degree, respectively, m, n, and $n-1$.

When expanded about the origin, $x=0$, the first non-vanishing derivative is then seen to be of order $m=n$. For $m>n$ the derivatives vanish alternately, beginning with $m=n+1$. As a consequence of these statements the following recursion formulae may be written:

$$a_{2k}D^{2k}(\sin x)_0^{2k} = f^{(2k)}(0) - \sum_{p=0}^{k-2} a_{2p+2}D^{2k}(\sin x)_0^{2p+2}, \qquad n \text{ even},$$

$$a_{2k+1}D^{2k+1}(\sin x)_0^{2k+1} = f^{(2k+1)}(0) - \sum_{p=0}^{k-1} a_{2p+1}D^{2k+1}(\sin x)_0^{2p+1}, \qquad n \text{ odd}.$$

Except for a_0, which is determined from the original equation, these formulae determine the a_n. To complete the expansion the usual methods of convergence must be applied.

Replacing $\sin x$ by $\cos x$ and expanding about $x=\pi/2$, similar formulae may be written to determine b_n in (2).

When n is even the recursion formula for expansion (1) may be replaced by

$$a_{2k}(2k)! = b_2 f''(0) + b_4 f^{IV}(0) + \cdots + b_{2k-2} f^{(2k-2)}(0) + f^{(2k)}(0),$$

where the b_{2i} are integers satisfying the equations:

$$\begin{aligned}
&-2^2 b_2 + 2^4 b_4 - 2^6 b_6 + \cdots + (-1)^{k-1} 2^{2k-2} b_{2k-2} + (-1)^k 2^{2k} = 0,\\
&-4^2 b_2 + 4^4 b_4 - 4^6 b_6 + \cdots + (-1)^{k-1} 4^{2k-2} b_{2k-2} + (-1)^k 4^{2k} = 0,\\
&-6^2 b_2 + 6^4 b_4 - 6^6 b_6 + \cdots + (-1)^{k-1} 6^{2k-2} b_{2k-2} + (-1)^k 6^{2k} = 0,\\
&\qquad\vdots \qquad\qquad\qquad\

A CERTAIN TYPE OF INTEGRAL

L. A. AROIAN, Hunter College

In a study of R. A. Fisher's z distribution [1] we found the semi-invariants and moments of z. From the mathematical point of view these statistical results may be used to evaluate

$$I(r, n_1, n_2) = \int_{-\infty}^{\infty} x^r y_x dx, \tag{1}$$

$$y_x = \frac{e^{n_1 x}}{(n_1 e^{2x} + n_2)^{(n_1+n_2)/2}}, \quad r, n_1, n_2, \text{ positive integers.}$$

We first give the semi-invariants of z, next find the moments of z about the origin from the semi-invariants, and finally show how (1) is related to the moments of z about the origin.

The ordinates of the z distribution are given by

$$y_z = k y_x, \qquad k = \frac{2 n_1^{n_1/2} n_2^{n_2/2}}{\mathrm{B}(n_1/2, n_2/2)}, \tag{2}$$

y_x as defined above, for $-\infty < z < \infty$. The semi-invariants of z are known [1]:

$$\lambda_{1:z} = \frac{1}{2}\log\frac{n_2}{n_1} - \frac{1}{2}\left\{\sum_{k=1}^{n_2/2-1}\frac{1}{k} - \sum_{k=1}^{n_1/2-1}\frac{1}{k}\right\}, \qquad n_1 \text{ and } n_2 \text{ even},$$

$$\lambda_{1:z} = \frac{1}{2}\log\frac{n_2}{n_1} - \left\{\sum_{k=0}^{(n_2-3)/2}\frac{1}{2k+1} - \sum_{k=0}^{(n_1-3)/2}\frac{1}{2k+1}\right\}, \qquad n_1 \text{ and } n_2 \text{ odd},$$

$$\lambda_{1:z} = \frac{1}{2}\log\frac{n_2}{n_1} + \frac{1}{2}\sum_{k=1}^{n_1/2-1}\frac{1}{k} - \sum_{k=0}^{(n_2-3)/2}\frac{1}{2k+1} + \sigma, \qquad n_1 \text{ even}, n_2 \text{ odd}, \tag{3}$$

$$\lambda_{1:z} = \frac{1}{2}\log\frac{n_2}{n_1} + \sum_{k=0}^{(n_1-3)/2}\frac{1}{2k+1} - \frac{1}{2}\sum_{k=1}^{n_2/2-1}\frac{1}{k} - \sigma, \qquad n_1 \text{ odd}, n_2 \text{ even},$$

$$\lambda_{r:z} = \sum_{k=0}^{\infty}(r-1)!\left\{\frac{(-1)^r}{(n_1+2k)^r} + \frac{1}{(n_2+2k)^r}\right\}, \qquad r \geqq 2.$$

$$\sigma = \log 2 = .693147181.$$

The moments about the origin of any distribution, μ_r', are related to the semi-invariants, λ_r, by the well-known formulas [2], [3]:

$$\begin{aligned}
\mu_1' &= \lambda_1 \\
\mu_2' &= \lambda_2 + \lambda_1^2 \\
\mu_3' &= \lambda_3 + 3\lambda_2\lambda_1 + \lambda_1^3 \\
\mu_4' &= \lambda_4 + 4\lambda_3\lambda_1 + 3\lambda_2^2 + 6\lambda_2\lambda_1^2 + \lambda_1^4
\end{aligned} \tag{4}$$

and in general

$$\mu_r' = \sum\sum\cdots\sum \frac{r!}{(a!)^u(b!)^v(c!)^w\cdots}\,\frac{\lambda_a^u\lambda_b^v\lambda_c^w\cdots}{u!v!w!\cdots},$$
$$a > b > c > \cdots,\ au + bv + cw + \cdots = r.$$

Since

$$\mu_r' = \int_{-\infty}^{\infty} z^r y_z dz = k\int_{-\infty}^{\infty} x^r y_z dx = kI(r, n_1, n_2),$$

we obtain our final result

$$I(r, n_1, n_2) = \frac{\mu_r'}{k}. \tag{5}$$

For μ_r' we substitute the results given by (4) and (3). For n_1 and n_2 large, approximate values of $\lambda_{r:z}$ may be used [1]. Special cases of (1) for particular values of n_1, n_2, and r may be found in Bierens de Haan [4, pp. 125, 127].

References

1. L. A. Aroian, A study of R. A. Fisher's z distribution and the related F distribution, Annals of Math. Stat., vol. 12, 1941 no. 4, pp. 429–448.

2. C. C. Craig, An application of Thiele's semi-invariants to the sampling problem, Metron vol. 7, 1928–29, pp. 3–74.

3. C. Jordan, Statistique Mathématique, Gauthier-Villars, Paris, 1927, p. 41.

4. Bierens de Haan, Nouvelles Tables d'Intégrales Définis, reprint, Stechert, New York, 1939.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Plane Trigonometry with Tables. By E. R. Heineman. New York, McGraw-Hill, 1942. 12+167+3+75 pages. $2.00.

The first sentence in the preface to this text states, "The principal objective of the author in writing this trigonometry textbook has been teachability." The reviewer feels that the author has clearly attained his objective.

There are many evidences throughout the text of close familiarity with the usual stumbling blocks that trip students of trigonometry and every effort is made to remove such obstructions. This is evidenced by such points as emphasis on the difference between cofunctions and reciprocal functions and warnings

against the use of the symbol "sin" without an angle written after it. Other details of development that make a favorable impression are (1) generous lists of selected problems including the true-false type, a total of over 100 problems on identities, (2) an approach to logarithms that emphasizes their meaning as well as their use in computation, (3) numerous illustrative examples giving model forms for solutions of problems. Although the author is unusually careful to explain the meaning of symbols and to give references, on page 17 the symbol for "less than" is employed in several problems without explanation.

In the general organization of material the text is to be commended for introducing the functions of general angles first and obtaining the functions of an acute angle as a special case. Reduction procedures are handled neatly and efficiently by a geometric discussion. The chapter on graphing the trigonometric functions is adequate but could easily be more complete by giving a discussion of the period, amplitude and sketch of functions of the type $y=A\cdot\sin(Bx+C)$. It is a question in the reviewer's mind whether the order of arrangement is advisable in presenting trigonometric equations before inverse functions. Furthermore there is no discussion of the general solution of trigonometric equations or of the general values of inverse functions. There is no treatment in the text itself of the use of small angles but a page in the tables gives a method of interpolating the value of the logarithm of the function of a small angle accurately to five places.

The fact remains that this is an extremely teachable textbook in plane trigonometry.

D. W. WESTERN

Mathematics in Daily Use. By W. W. Hart, Cottell Gregory and Veryl Schult. Boston, D. C. Heath and Company, 1942. 7+376 pages. $1.32.

This text is designed for students in secondary schools who for various reasons will not study algebra or for students who will study algebra later. It is not intended to replace first-year algebra.

The material covered is rather complete for such a text. It includes integers, fractions, measurements, percentage, graphs, family income, community activities, business transactions, and an introduction to algebra. A student who masters the material included in this text will have a good working knowledge of problems encountered in everyday life.

Fractions, both common and decimal, and percentage are fully treated. Illustrative problems are good, and examples are many and varied. Insurance, taxes, installment buying, savings, interest, margin, profit—all topics of everyday business—are explained.

Many students will enjoy the chapter on *Leisure Time Problems*, especially the puzzles and games of chance.

Short tests on each chapter are included, and there is an ample supply of examples and problems for class use and homework.

SARA L. NELSON

Principles of College Algebra. By M. S. Knebleman and T. Y. Thomas. New York, Prentice-Hall, 1942. 10+380 pages. $2.50.

The authors have as one of their purposes that of teaching the student that algebra is based on reason and not on magic. To this end they here develop the subject from a set of postulates and definitions which they insist that the student thoroughly understand. Lest the beginners think that they are being hoodwinked at the start, the Dedekind Cut Postulate is carefully explained in the first chapter where the real number system is built up. The fifteen chapters give a careful treatment of the usual topics of college algebra including partial fractions, logarithms, matrices and determinants, the theory of equations, permutations and combinations, probability, and infinite sequences and series.

The exercises are numerous and well graded. Problems of the "story" type are almost entirely lacking "since these involve non-algebraic or superficial difficulties that tend to disappear with general mental growth and increased familiarity with the subject." Besides the exercises after each topic there are cumulative exercises at the end of each chapter. These have been selected, for the most part, from recent college examinations. Careful references to the sources are given so that the students will be fully aware, for example, that the question "Define: (a) irrational number, (b) real number" (page 29) was taken from an examination of Arizona University. The book concludes with ten complete final examinations given by a number of unspecified colleges and universities of wide geographical distribution. Answers are given to most of the problems in the exercises, but not to the problems on the final examinations.

D. W. HALL

NEW BOOKS RECEIVED

Analytic Geometry. By E. S. Smith, M. Salkover, and H. K. Justice. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1943. 12+298 pages. $2.50.

Basic Mathematics for Pilots and Flight Crews. By C. V. Newsom and H. D. Larsen. New York, Prentice-Hall, Inc., 1943. 6+153 pages. $2.00.

Elementary Navigation. By L. E. Moore. Boston, D. C. Heath & Co., 1943. 7+222 pages. $1.60.

Arithmetic for the Emergency. By G. M. Ruch, F. B. Knight, and J. W. Studebaker. Chicago, Scott, Foresman and Company, 1942. 170 pages. $0.64.

Mathematics for the Emergency. By C. J. Lapp, F. B. Knight, and H. L. Rietz. Chicago, Scott, Foresman & Company, 1942. 158 pages $1.80.

Formalization of Logic. By Rudolf Carnap. Cambridge, Harvard University Press; London, Humphrey Milford, 1943. 18+159 pages. $3.00.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 576. *Proposed by M. A. Dernham, San Francisco*

What well known type of sequence may be defined as one in which each term but the first may be obtained by taking the preceding term and a common constant and dividing their product by their sum?

E 577. *Proposed by V. Thébault, San Sebastián, Spain*

Given an "isosceles" tetrahedron $A_1A_2A_3A_4$ (so that every two opposite edges are equal), let perpendiculars be drawn to the faces $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ at their circumcenters O_1, O_2, O_3, O_4, to meet the hemispheres described exteriorly (or interiorly) on the respective circumcircles in P_1, P_2, P_3, P_4. Show that the tetrahedra $O_1O_2O_3O_4$ and $P_1P_2P_3P_4$ are isosceles, and that they have the same centroid as $A_1A_2A_3A_4$.

E 578. *Proposed by R. V. Heath, Wall St., New York City*

Find a perfect cube whose digits form a permutation of consecutive digits. (Cf. E 538.)

E 579. *Proposed by Howard Eves, Syracuse University*

Show that a matrix of m rows and n columns contains $(2^m-1)(2^n-1)$ submatrices.

E 580. *Proposed by Albert Furman, Infantry School, Fort Benning*

A closed tank containing V gallons of water at temperature T has an inlet pipe which supplies water at temperature t. Assuming an ideal situation where there is no loss of heat and an instantaneous diffusion in the mixture, show that the temperature of v gallons of water drawn from the tank into an open container is

$$t + (T - t)(1 - e^{-v/V})V/v.$$

SOLUTIONS

Three Similar Polygons

E 451 [1942, 613]. *Proposed by Joseph Rosenbaum, Bloomfield, Conn.*

Given a regular polygon of n sides, $n>4$, design a quadrilateral, Q, such that (1) it shall be possible to fit $2n$ of the Q's to the polygon to form a new regular polygon of n sides, and (2) it shall be possible to fit $2n$ additional Q's to the new polygon to form a still larger third regular polygon of n sides.

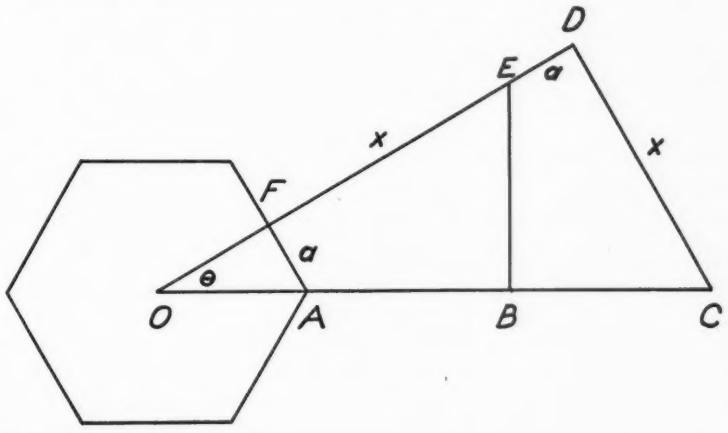
Solution by Howard Eves, Syracuse University. Let $2a$ be the side of the given polygon, and let $\theta=180°/n$. Then the problem will be solved if we can construct two congruent quadrilaterals $FEBA$ and $DCBE$ (see figure). Set $FE=DC=x$. Then we must have

$$x = DC = DO \tan\theta = (a + x + a\cot\theta)\tan\theta = x\tan\theta + a(1+\tan\theta),$$

whence

$$x = a(1+\tan\theta)/(1-\tan\theta).$$

This will always be possible if $\tan\theta<1$, *i.e.*, if $\theta<45°$, or $n>4$.



Also solved by the proposer.

Isogonal Conjugates

E 537 [1942, 546]. *Proposed by V. Thébault, San Sebastián, Spain*

Let L, M, N and L', M', N' be the orthogonal projections of a point P on the sides and corresponding altitudes of a given triangle. Show that the lines LL', MM', NN' are in general concurrent, and find the locus of P when they are parallel.

Solution by J. H. Butchart, Grinnell College. Let AD, BE, CF be the altitudes of the given triangle ABC, and let U, V, W be the midpoints of PD, PE, PF.

Then U is the center of the rectangle $PLDL'$. The line through D parallel to LL', being the image of DP by reflection in AD, is the isogonal conjugate of DP with respect to the orthic triangle DEF. This, and the two analogous lines through E and F, meet in the isogonal conjugate point to P with respect to DEF. Hence the lines LL', MM', NN' for the homothetic figure UVW are likewise concurrent.

The isogonal conjugate lines for PD, PE, PF are parallel only when P lies on the circumcircle of DEF, which is the nine-point circle of ABC. This, therefore, is the locus of P when LL', MM', NN' are parallel.

Also solved by W. B. Carver, Howard Eves, L. M. Kelly, and the proposer.

Eves generalizes this to an *affine* problem, replacing the altitudes by any three Cevians HA, HB, HC. He projects P parallel to the Cevians onto the sides, and *vice versa* (so that Butchart's rectangles become parallelograms). Instead of the circle we now have a "nine-point conic": the locus of centers of conics through the four points A, B, C, H. (When the quadrangle $ABCH$ is convex, this locus is a hyperbola whose asymptotes are diameters of the two circumscribed parabolas. See H. F. Baker, *Principles of Geometry*, Cambridge, Eng., 1930, p. 41: "The eleven-point conic.")

The proposer remarks that (in the original problem) the lines LL', MM', NN' are diameters of a conic inscribed in the triangle DEF. The foci of this conic are P and the isogonal conjugate point. When the diameters are parallel, the conic must be a parabola, and its focus P lies on the circumcircle of DEF.

The Centers of Three Circles

E 539 [1942, 546]. *Proposed by Howard Eves, Syracuse University*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

Elucidation by the Proposer. We define a ruler construction as any construction carried out by the straight-edge alone under Euclid's first two postulates:

I. We may connect any two given points by a straight line.

II. We may extend any given straight line indefinitely in either direction.

The Poncelet-Steiner Theorem states that in the presence of a circle *and its center* we may carry out any Euclidean construction with ruler alone. This naturally suggests the question: How many circles do we need in the plane in order to find their centers by a ruler construction? In this connection we have the following results:

(a) If we are given one circle alone, the ruler is *not* sufficient to find its center.

(b) If we are given two circles which intersect, are tangent, or are concentric, the ruler *is* sufficient to find their centers.

(c) If the two circles are non-intersecting, non-tangent, and non-concentric, then the ruler is *not* sufficient to find their centers, but we need a third circle.

For (a) and part of (b), see, *e.g.*, H. Steinhaus, *Mathematical Snapshots*, New York, 1938, pp. 44, 39. Result (c) is established by Cauer, Schur, and Mieren-

dorff, *Mathematische Annalen*, vol. 73 (1912), pp. 90–94 and vol. 74 (1913), pp. 462–464. However, no neat and direct construction is offered for the case of three circles.

Editorial Note. Contributions on this subject will still be welcomed.

The Sixteen-Point Sphere

E 540 [1942, 546]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

Can the radius of the sixteen-point sphere ever be one-half the circumradius of the tetrahedron? (The sixteen-point sphere passes through the circumcenters of the faces.)

Solution by Howard Eves, Syracuse University. We answer the question in the affirmative and back the assertion by considerations of continuity, as follows. Let R be the circumradius, and r the radius of the sixteen-point sphere. If the tetrahedron is regular, then $r=R/3$; and if it is trirectangular, then $r=\infty$ with R finite. Since we may continuously deform a regular tetrahedron into a trirectangular one, it follows that at some intermediate stage we must have $r=R/2$. (Incidentally, it must likewise be possible to have $r=R$.)

A Square with Four Equal Digits

E 542 [1942, 613]. *Proposed by V. Thébault, San Sebastián, Spain*

In what scale of notation (with radix less than a hundred) will the four-digit number 58 58 58 58 be a perfect square?

Solution by Irving Kaplansky, Harvard University. Let the radix be r, so that $58<r<100$. For $n=58(r+1)(r^2+1)$ to be a square, r must be odd, as otherwise n would be twice an odd number. Also either $r+1$ or r^2+1 must be divisible by 29. Since the solution of

$$r^2 \equiv -1 \pmod{29}$$

is $r\equiv\pm 12$, we have $r\equiv -1$ or $\pm 12 \pmod{29}$. These conditions are fulfilled only by $r=75$ or 99. The former is rejected by trial, or perhaps more neatly by observing that with $r=75$ we would have $n\equiv 3 \pmod 5$, a non-residue. With $r=99$ we have

$$\sqrt{n} = 7540 \text{ (denary)}$$
$$= \overline{76}\ \overline{16} \text{ (in the scale of 99).}$$

Also solved by D. H. Browne, R. C. Buck, W. E. Buker, E. P. Starke, and the proposer.

Perpendicular Polar Planes

E 543 [1942, 613]. *Proposed by N. A. Court, University of Oklahoma*

Find a point whose polar planes for three given spheres (with non-collinear centers) are mutually perpendicular. Show that the problem may have two solutions. When will they be real?

Solution by R. C. Buck, Cambridge, Mass. Let P be the desired point, and P_1, P_2, P_3 the centers of the spheres. Then it is necessary and sufficient that the lines PP_1, PP_2, PP_3 be mutually perpendicular. The locus of points such that PP_i is perpendicular to PP_j is the sphere with diameter P_iP_j. By considering the possible intersections of three such spheres, we see that there are two real points P, or none, according as the triangle $P_1P_2P_3$ is acute or obtuse. In the intermediate case of a right triangle, there is just one point P, but one of its polar planes is the plane at infinity.

Also solved by Howard Eves and the proposer.

A Commensurable Tetrahedron

E 544 [1942, 613]. *Proposed by E. P. Starke, Rutgers University*

Show that it is possible to construct a tetrahedron such that the length of every edge, the area of every face, and the volume all are integers.

Solution by the Proposer. Such a tetrahedron (probably not the simplest) has one edge 896, the opposite edge 990, and the remaining four each 1073. The areas of the faces are 436800 and 471240 (two each), and the volume is 62092800.

To determine this example, a rhombus $PQRS$ was taken with each side equal to c, and diagonals $PR=2a$ and $QS=2b$, whence

$$a^2 + b^2 = c^2. \tag{1}$$

The triangle PRS was rotated about PR until the distance QS became $2a'$. It is required that a', a, b, c be integers, and also that there exist integers b' and k such that

$$a'^2 + b'^2 = c^2, \qquad k^2 + a'^2 = b^2. \tag{2}$$

The tetrahedron $PQRS$ will then have edges $2a$, $2a'$, c, c, c, c; face areas ab, ab, $a'b'$, $a'b'$; and volume $2aa'k/3$.

To solve the equations (1) and (2), we note that c^2 is a sum of two squares in two ways. Thus c is a product of two integers each of which is a sum of two squares. This suggests the equations

$$u^2 = r^2 + s^2, \qquad u'^2 = r'^2 + s'^2, \qquad c = uu',$$

$$a = |rr' - ss'|, \qquad b = rs' + r's, \qquad a' = |r's - rs'|, \qquad b' = rr' + ss',$$

$$k^2 = 4rr'ss'.$$

By convassing the simplest solutions of the Pythagorean equation $u^2=r^2+s^2$, the following two are found for which $rr'ss'$ is a square:

$$20^2 + 21^2 = 29^2, \qquad 35^2 + 12^2 = 37^2.$$

The values $r=20$, $r'=35$, $s=21$, $s'=12$, $u=29$, $v=37$, thus suggested, determine the tetrahedron given above.

Concyclic Points on the Sides of a Triangle

E 545 [1942, 614]. *Proposed by A. H. Stone, Institute for Advanced Study*

Starting with a point P on the side BC of a triangle ABC, mark Q on AB with $BQ=BP$, R on CA with $AR=AQ$, P' on BC with $CP'=CR$, Q' on AB with $BQ'=BP'$, and so on. Prove that the construction closes, *i.e.*, that $CP=CR'$, and that the six points P, Q, R, P', Q', R' are concyclic. (Some obvious restrictions must be placed on the directions on the sides of the triangle in which the intervals BQ, *etc.*, are taken.)

Solution by Howard Eves, Syracuse University. Let P_0, Q_0, R_0 be the points of contact of the incircle. Then we see that

$$PP_0 = QQ_0 = RR_0 = P'P_0 = Q'Q_0 = R'R_0 = P''P_0.$$

This shows that the construction closes, and that P, Q, R, P', Q', R' lie on a circle concentric with the incircle.

Note. This proof generalizes to any odd polygon possessing an incircle.

Also solved by R. K. Allen, W. B. Clarke, Joseph Rosenbaum, Alan Wayne, M. W. White, and the proposer. Concerning the generalization to any odd polygon, Rosenbaum remarks that the construction may close even if there is no incircle. For, given a polygon, it may be possible to make the sides touch a circle, without altering their lengths, by suitably changing the angles. This process will not affect the positions of the points P, Q, $\cdots$ on their respective sides.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4086. *Proposed by P. Erdös, Princeton, N. J.*

Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon, and O any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation.

$$\pi\left(1 - \frac{1}{2n+1}\right) \leqq A_iOA_j < \pi.$$

4087. *Proposed by Betty Dick and B. M. Stewart, Michigan State College*

Let P be a plane polygon with vertices $A_1, A_2, \cdots, A_n$, and consider $A_{n+k}=A_k$. Determine points $B_1, B_2, \cdots, B_n$ such that B_i is on the side A_iA_{i+1}

with $A_iA_{i+1}=k\cdot A_iB_i$, where k is a fixed real number. Let the lines A_iB_{i+1} and $A_{i+1}B_{i+2}$ intersect in the point C_i, thus determining the polygon Q with the vertices $C_1, C_2, \cdots, C_n$. Let $R(k)$ be the ratio of the area of Q to the area of P, wherein not to restrict the type of polygon we use Klein's definition of area, *Elementary Mathematics from an Advanced Standpoint, Geometry*, p. 10. (1) Show that k and $R(k)$ are invariants under affine transformations. (2) As a corollary to (1) show that for any triangle we have $R(k)=(k-2)^2/(k^2-k+1)$. (3) For any parallelogram we have $R(k)=(k-1)^2/(k^2+1)$. (4) Show that $R(k)$ does not have the same value for all quadrilaterals.

This problem is a generalization of the so-called Problem of Steinhaus which asserts for any triangle $R(3)=1/7$. An early reference, suggestive of this problem, is problem 276 in *Mathesis*.

4088. *Proposed by V. Thébault, San Sebastián, Spain*

If three circles $A(\rho)$, $B(\rho)$, $C(\rho)$ with the same radius ρ are described in the triangle ABC, and then the circles with centers A, B, C orthogonal respectively to $C(\rho)$, $A(\rho)$, $B(\rho)$; these three circles have the same radical center M_1 whatever the value of ρ. The same is true of three circles with centers A, B, C orthogonal respectively to $B(\rho)$, $C(\rho)$, $A(\rho)$, the radical center being M_2. If O is the circumcenter of ABC, show also that: (1) The triangles ABC and OM_1M_2 have the same centroid; (2) The straight line M_1M_2 is perpendicular to the straight line through the centroid and the Lemoine point; (3) if M_1' and M_2' are the symmetrics of M_1 and M_2 with respect to O, then M_1M_1' and M_2M_2' are parallel to the Euler line.

SOLUTIONS

Ellipses Inscribed in a Square

4029 [1942, 202]. *Proposed by H. L. Dorwart, Washington and Jefferson College*

Let d_1, d_2, d_3, d_4 be the distances in order from the sides of a square of length k units to any interior point P. Then

$$(\sqrt{d_1d_2}\pm\sqrt{d_3d_4})/k \quad\text{and}\quad (\sqrt{d_1d_4}\pm\sqrt{d_2d_3})/k$$

represent the sines and cosines of two angles θ_1 and θ_2, since the sum of the squares of these expressions equals

$$(d_1d_2+d_3d_4+d_1d_4+d_2d_3)/k^2=(d_1+d_3)(d_2+d_4)/k^2=1.$$

Give a geometric interpretation for the angles θ_1 and θ_2.

Solution by the Proposer. To be specific, let

$$(1)\qquad \begin{cases}\sin\theta_1=(\sqrt{d_1d_2}+\sqrt{d_3d_4})/k, & \cos\theta_1=(\sqrt{d_1d_4}-\sqrt{d_2d_3})/k\\ \sin\theta_2=(\sqrt{d_1d_2}-\sqrt{d_3d_4})/k, & \cos\theta_2=(\sqrt{d_1d_4}+\sqrt{d_2d_3})/k,\end{cases}$$

and let the vertices of the square be $A(K, O)$, $B(O, K)$, $C(-K, O)$, $D(O, -K)$, where $\sqrt{2}\cdot K=k$. Then consider the family of conics

$$\frac{x^2}{(1-\lambda^2)K^2}+\frac{y^2}{\lambda^2K^2}=1, \tag{2}$$

which for $\lambda^2<1$ are ellipses inscribed in the square.

Writing (2) as a quadratic equation in λ^2,

$$K^2\lambda^4+(x^2-y^2-K^2)\lambda^2+y^2=0,$$

we obtain

$$4K^2=(y+K+x)(y+K-x)+(y-K+x)(y-K-x)$$
$$\pm 2\sqrt{(x-y+K)(x+y-K)(x-y-K)(x+y+K)}.$$

This can be condensed if we put

$$l_1=-x-y+K, \qquad l_3=x+y+K,$$
$$l_2=x-y+K, \qquad l_4=-x+y+K,$$

whence

$$4K^2\lambda^2=l_1l_2+l_3l_4\pm 2\sqrt{l_1l_2l_3l_4}$$

and

$$\pm 2K\lambda=\sqrt{l_1l_2}\pm\sqrt{l_3l_4}.$$

also

$$4K^2(1-\lambda^2)=4K^2-(l_1l_2+l_3l_4)\mp 2\sqrt{l_1l_2l_3l_4}.$$

But $(l_1+l_3)(l_2+l_4)=4K^2$, hence

$$4K^2(1-\lambda^2)=l_1l_4+l_2l_3\mp 2\sqrt{l_1l_2l_3l_4}$$

and

$$\pm 2K\sqrt{1-\lambda^2}=l_1l_4\mp l_2l_3.$$

Evidently $l_1=\sqrt{2}d_1$, $l_2=\sqrt{2}d_2$, $l_3=\sqrt{2}d_3$, $l_4=\sqrt{2}d_4$, so that

$$\pm\lambda=(\sqrt{d_1d_2}\pm\sqrt{d_3d_4})/k$$

and

$$\pm\sqrt{1-\lambda^2}=(\sqrt{d_1d_4}\mp\sqrt{d_2d_3})/k.$$

Using (1), it is now evident that

$$\pm\frac{\lambda}{\sqrt{1-\lambda^2}}=\tan\theta_1 \text{ or } \tan\theta_2.$$

In general, through any point P inside the square, there will pass two members of (2), and if for the ellipse $x^2/a^2+y^2/b^2=1$ we call the lines $y=\pm bx/a$ the *diagonals* of the ellipse, we can now interpret θ_1 and θ_2 as the inclinations of the

diagonals of the two members of the inscribed family of ellipses passing through any general point P of the square $ABCD$.

Editorial Note. The only other conics tangent to the sides of the square are two types of hyperbolas whose focal axes are respectively on the diagonals AC and BD; and the only real intersections P of these hyperbolas are those for two of the same type. Hence P must lie within the vertical angles at the vertices, for example one such point P is within the angle between the extensions of BA and DA. If P is within this angle and the distances $d_i > 0$ are defined as above, we may write

$$(1)\qquad \begin{aligned} k\cosh\theta_1 &= \sqrt{d_2d_3} + \sqrt{d_4d_1}, & k\sinh\theta_1 &= \sqrt{d_1d_2} + \sqrt{d_3d_4},\\ k\cosh\theta_2 &= \sqrt{d_2d_3} - \sqrt{d_4d_1}, & k\sinh\theta_2 &= \sqrt{d_1d_2} - \sqrt{d_3d_4}, \end{aligned}$$

and from these we obtain

$$(2)\qquad \begin{aligned} d_1 &= k\sinh^2\left(\frac{\theta_1+\theta_2}{2}\right), & d_2 &= k\cosh^2\left(\frac{\theta_1-\theta_2}{2}\right),\\ d_3 &= k\cosh^2\left(\frac{\theta_1+\theta_2}{2}\right), & d_4 &= k\sinh^2\left(\frac{\theta_1-\theta_2}{2}\right). \end{aligned}$$

It then follows from the equations of the sides that the coordinates of P are

$$(3)\qquad P:\ (k/\sqrt{2})\cosh\theta_1\cosh\theta_2, \qquad (k/\sqrt{2})\sinh\theta_1\sinh\theta_2.$$

This suggests at once the equations of two hyperbolas of the same type

$$(4)\qquad H_i:\ x = (k/\sqrt{2})\cosh\theta_i\cosh\theta, \qquad y = (k/\sqrt{2})\sinh\theta_i\sinh\theta, \qquad i = 1, 2.$$

If $\theta=\theta_2$ in H_1 we see that P lies on H_1; if $\theta=\theta_1$ in H_2 we see that it lies also on H_2. If in H_1 we set $\theta=-\theta_1$ we easily find that the equation of AB, $x+y-k/\sqrt{2}=0$ is satisfied. The slope of H_1 at the point θ is $\tanh\theta_1/\tanh\theta$, and hence H_1 is tangent to AB at the point for $\theta=-\theta_1$; similarly for H_2 at $\theta=-\theta_2$. Thus the points of contact T_{1i} of H_i with AB have the coordinates

$$(4)\qquad T_{1i}:\ (k/\sqrt{2})\cosh^2\theta_i, \qquad -(k/\sqrt{2})\sinh^2\theta_i;$$

and the straight lines $y=\pm\tanh^2\theta_i x$ meet the sides of the square in the four points of contact of H_i with the sides. If the two distinct points T_{11} and T_{12} are given on the extension of BA as points of contact of H_1 and H_2 then the corresponding point P of intersection has coordinates which are the geometric means of the coordinates of the two given points on BA using suitable signs for the y's. The two hyperbolas have slopes at each of their four intersections whose product is unity.

For the case of the ellipses where P is inside the square we may write

$$\begin{aligned} k\cos\theta_1 &= \sqrt{d_2d_3} - \sqrt{d_4d_1}, & k\sin\theta_1 &= \sqrt{d_3d_4} + \sqrt{d_1d_2},\\ k\cos\theta_2 &= \sqrt{d_2d_3} + \sqrt{d_4d_1} & k\sin\theta_2 &= \sqrt{d_3d_4} - \sqrt{d_1d_2}, \end{aligned}$$

so that if P is within the triangle OAB the four functions are each positive. It then follows that

$$d_1 = k \sin^2\left(\frac{\theta_1 - \theta_2}{2}\right), \qquad d_2 = k \cos^2\left(\frac{\theta_1 + \theta_2}{2}\right),$$

$$d_3 = k \cos^2\left(\frac{\theta_1 - \theta_2}{2}\right), \qquad d_4 = k \sin^2\left(\frac{\theta_1 + \theta_2}{2}\right);$$

and P has the coordinates $(k/\sqrt{2}) \cos\theta_1 \cos\theta_2$, $(k/\sqrt{2}) \sin\theta_1 \sin\theta_2$. We then obtain similar results in a similar manner.

Arithmetic Progressions in Positive Integers

4031 [1942, 262]. *Proposed by S. H. Gould, Victoria University, Toronto*

Let m be any fixed positive integer, $k=1, 2, 3, \cdots$, and $r=0, 1, 2, \cdots, m(k-1)$. Take unity for first term when $k=1$, construct inductively the arithmetic progression of order $m(k-1)$, the first term of whose rth difference series is the $(r+1)$st term of the A.P. of order $m(k-2)$. Prove that its $(mk+2)$nd term is k^{mk}.

Solution by the Proposer. Letting A_k^i denote the $(i+1)$st term of the A.P. of order $m(k-1)$, it is clear that we must prove that, if

$$A_1^i = 1 \qquad i = 0, 1, 2, \cdots$$

$$A_k^i = \sum_{s=0}^{\min[i, m(k-1)]} \binom{i}{s} A_{k-1}^s \qquad k = 2, 3, \cdots \tag{1}$$

then

$$A_k^{mk+1} = k^{mk}.$$

Writing $(mk+1) = s_1$ and $s = s_2$, we have

$$A_k^{mk+1} = \sum_{s_2=0}^{s_1}{}' \binom{s_1}{s_2} \cdot A_{k-1}^{s_2} = \sum_{s_2=0}^{s_1}{}' \cdot \sum_{s_3=0}^{s_2}{}' \binom{s_1}{s_2} \cdot \binom{s_2}{s_3} \cdot A_{k-2}^{s_3}$$

$$= \cdots = \sum_{s_2=0}^{s_1}{}' \sum_{s_3=0}^{s_2}{}' \cdots \sum_{s_k=0}^{s_{k-1}}{}' \binom{s_1}{s_2}\binom{s_2}{s_3} \cdots \binom{s_{k-1}}{s_k}, \tag{2}$$

$$= \sum_{s_1 s_2 \cdots s_k}{}' \frac{s_1!}{(s_1 - s_2)!(s_2 - s_3)! \cdots s_k!} = \sum_{\lambda_1 \cdots \lambda_k}{}' \frac{(mk+1)!}{\lambda_1! \lambda_2! \cdots \lambda_k!},$$

where $\lambda_1 = (s_1 - s_2)$ *etc.*, and the primes indicate that summations are to be made only for:

$$s_t = \sum_{j=t}^{k} \lambda_j \leqq m(k+1-t) \qquad t = 2, 3, \cdots, k. \tag{B}$$

By the polynomial theorem, $k^{mk+1} =$ (1) without the restriction (B), and (see

lemma), among the k cyclic permutations of any set $(\lambda_1, \lambda_2, \cdots, \lambda_k)$, there is exactly one which satisfies (B).

Thus $A_k^{mk+1} = k^{mk}$.

LEMMA. *For a given ordered set* $\lambda_1, \lambda_2, \cdots, \lambda_k$, $k \geqq 2$, *and its cyclic permutations, of non-negative integers whose sum is* $mk+1$, *where* m *is a positive integer, there is among these* k *permutations one and only one set (i.e., there is exactly one value of* a*) for which*

$$\sum_{j=a+1}^{a+q} \lambda_j \leqq mq, \qquad q = 1, 2, \cdots, k-1. \tag{3}$$

This is equivalent to (B).

Proof. There cannot be two sets satisfying (3). For, if there are two such distinct sets, say for $a = 0$ for one and $1 \leqq a < k$ for the other; then $\lambda_1 + \lambda_2 + \cdots + \lambda_a \leqq ma$ and $\lambda_{a+1} + \lambda_{a+2} + \cdots + \lambda_k \leqq m(k-a)$, and the sum gives the contradiction $mk+1 \leqq mk$.

Suppose now that no set satisfies (3); we shall deduce in this case the contradiction that the sum of the k integers λ_j is greater than $mk+1$. To do this, we let $v_1, v_2, \cdots, v_i$ be the smallest positive integers such that

$$\sum_{j=1}^{v_1} \lambda_j > mv_1, \quad \sum_{j=v_1+1}^{v_1+v_2} \lambda_j > mv_2, \cdots, \quad \sum_{j=v_1+v_2+\cdots+v_{i-1}+1}^{v_1+v_2+\cdots+v_i} \lambda_j > mv_i, \tag{4}$$

with $v_1 + v_2 + \cdots + v_{i-1} < k \leqq v_1 + v_2 + \cdots + v_i = k + l$. This means that, if $l > 0$, the last element λ_l of the subset (v_i) in (4) falls in a prior subset (v_r), $r \geqq 1$, and we first suppose that λ_l does not fall at the end of (v_r). Then we have from the inequalities in (4) for (v_r)

$$\lambda_{v_1+v_2+\cdots+v_{r-1}+1} + \cdots + \lambda_l + \lambda_{l+1} + \cdots + \lambda_{v_1+v_2+\cdots+v_r} > mv_r,$$

and from the minimal choice of v_r

$$\lambda_{v_1+v_2+\cdots+v_{r-1}+1} + \cdots + \lambda_l \leqq m[l - (v_1 + v_2 + \cdots + v_{r-1})],$$

from which it follows that

$$\sum_{j=l+1}^{v_1+v_2+\cdots+v_r} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l). \tag{5}$$

Also by addition of the inequalities in (4) for (v_{r+1}), (v_{r+2}), $\cdots$, (v_{i-1}), we have

$$\sum_{j=v_1+v_2+\cdots+v_r+1}^{v_1+v_2+\cdots+v_{i-1}} \lambda_j > m(v_{r+1} + v_{r+2} + \cdots + v_{i-1}). \tag{6}$$

The upper limit for the sum of (v_i) in (4) may be written $k+l$, and by addition of this inequality with (5) and (6) we have

$$(7) \qquad \sum_{j=l+1}^{l+k} \lambda_j > m(v_1 + v_2 + \cdots + v_r - l + v_{r+1} + \cdots + v_i) + 2 = mk + 2.$$

If λ_l falls at the end of (v_r), then (5) drops out and we have again a contradiction. If $l=0$, the sum of the inequalities in (4) gives at once the contradiction that the sum of the k integers λ_j is $\geqq mk+2$, $i \geqq 2$.

Editorial Note. This interesting problem and its interesting solution were received from R. D. James with the following account of the origin of the problem. In an article by Heilbronn, Landau, and Sherk in the *Journal Tchécoslovaque de Mathématique et de Physique*, 65, 1935–36, pp. 117–140, there is a lemma (*Satz* 8) equivalent to the following: Given the numbers A_k^t defined as in (1) above but for $m=2$, then $A_k^{2k+1} \leqq k^{2k}$. After a study of this it seemed to James that the result should be an actual equality, but he could not find a proof and suggested the problem to J. S. Vigder. The latter considered the more general problem using the positive integer m in place of 2, and saw that a proof involved the polynomial theorem as above, but he was unable to complete the proof and passed the matter on to the proposer. The proposer formulated the problem differently and came through with a solution resulting from his proof of his lemma (3).

An Oval and its Normal Expansion

4036 [1942, 340]. *Proposed by L. A. Santaló, Rosario, Argentina*

Let C_1 be an oval with a continuously varying radius of curvature R; at each point of C_1 a normal of length R is drawn exteriorly giving points of a second curve C_2 (which may not be convex); and let A be the area enclosed between the two curves. From a chosen fixed point a vector is drawn parallel to the normal at a point of C_1 and of length R for that point, thus giving as the point varies on C_1 a curve C_3 having the area A_3 and length L_3. If L_2 is the length of C_2 and A_1 is the area of C_1, show that

$$\text{(a)} \quad A = 3A_3; \qquad \text{(b)} \quad L_2L_3 \geqq 8\pi A_1;$$

the equality in (b) is true only when C_1 is a circle.

I. *Solution by Fritz John, University of Kentucky.* Let $p(\alpha)$ denote the "function of support" of C_1, *i.e.*, $p(\alpha)$ shall be the distance of that tangent of C_1 from the origin, whose normal forms the angle α with the x-axis (See Courant: *Calculus*, II, p. 213). Then

$$x = p\cos\alpha - p'\sin\alpha, \qquad y = p\sin\alpha + p'\cos\alpha$$

is a parametric representation for C_1. The radius of curvature of C_1 is given by $R=p+p''$, the enclosed area by

$$A_1 = \frac{1}{2}\int_0^{2\pi} (xy' - yx')d\alpha = \frac{1}{2}\int_0^{2\pi} (p^2 + pp'')d\alpha = \frac{1}{2}\int_0^{2\pi} (p^2 - p'^2)d\alpha.$$

Similarly the parametric representations of C_2 and C_3 are respectively

$$x = (2p + p'') \cos \alpha - p' \sin \alpha, \qquad y = (2p + p'') \sin \alpha + p' \cos \alpha,$$

and

$$x = (p + p'') \cos \alpha, \qquad y = (p + p'') \sin \alpha;$$

hence the areas enclosed by C_2 and C_3 are easily found to be

$$A_2 = \frac{1}{2} \int_0^{2\pi} (4p^2 - 7p'^2 + 3p''^2) d\alpha$$

$$A_3 = \frac{1}{2} \int_0^{2\pi} (p^2 - 2p'^2 + p''^2) d\alpha.$$

Consequently $A = A_2 - A_1 = 3A_3$, which is the first statement.

Now "Wirtinger's inequality" states, that for a function $f(\alpha)$ of class C' with $\int_0^{2\pi} f(\alpha) d\alpha = 0$

$$\int_0^{2\pi} f'^2(\alpha) d\alpha > \int_0^{2\pi} f^2(\alpha) d\alpha,$$

unless f is of the form $f(\alpha) = a \cos \alpha + b \sin \alpha$; (see Hardy-Littlewood-Polya: *Inequalities*, pp. 185–187). For $f = p'$ it follows that $\int_0^{2\pi} p''^2(\alpha) d\alpha > \int_0^{2\pi} p'^2(\alpha) d\alpha$, and hence $A_3 > A_1$, unless $p' = a \cos \alpha + b \sin \alpha$; in the latter case $p = c + a \sin \alpha - b \cos \alpha$, and C_1 is a circle of radius c. The isoperimetric inequality (which may be based on Wirtinger's inequality), yields

$$L_2^2 \geqq 4\pi A_2, \qquad L_3^2 \geqq 4\pi A_3;$$

hence

$$L_2 L_3 \geqq 4\pi \sqrt{A_2 A_3} = 4\pi \sqrt{(3A_3 + A_1) A_3} > 4\pi \sqrt{4A_1^2} = 8\pi A_1$$

unless C_1 is a circle.

In the case where C_1 is a circle of radius c, C_2 is a circle of radius $2c$, and C_3 a circle of radius c, so that $L_2 L_3 = 8\pi A_1$.

II. *Solution by the Proposer.* We consider two normals to C_1 corresponding to the directions ϕ and $\phi + d\phi$; a point on a normal whose distance to C_1 is a constant equal to λ will describe a curve whose arc s^* satisfies

$$ds^* = (R + \lambda) d\phi.$$

The area A will be then

$$A = \iint ds^* d\lambda = \int_0^{2\pi} d\phi \int_0^R (R + \lambda) d\lambda = \frac{3}{2} \int_0^{2\pi} R^2 d\phi = 3A_3$$

which proves (a).

We have also, if s_2 is the arc of C_2,

(1) $$ds_2 = \sqrt{4R^2 d\phi^2 + dR^2} = \sqrt{4R^2 + R'^2}\, d\phi$$

where R' represents the derivative with respect to ϕ. We have also

(2) $$ds_3 = \sqrt{R^2 + R'^2}\, d\phi.$$

From (1) and (2) we deduce, representing by s_1 the arc of C_1

$$ds_2 \geqq 2Rd\phi = 2ds_1 \quad \text{and} \quad L_2 \geqq 2L_1$$
$$ds_3 \geqq Rd\phi = ds_1 \quad \text{and} \quad L_3 \geqq L_1.$$

This gives us

(3) $$L_2 L_3 \geqq 2L_1^2$$

But it is known that for every plane closed curve we have $L_1^2 - 4\pi A_1 \geqq 0$; so this inequality and (3) proves the last part (b).

The equality in (b) is valid only if $R' = 0$, and then the radius of curvature is constant and the closed curve must be a circle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. H. F. Bright of San Angelo College has been appointed to an assistant professorship at Denison University.

Dr. Jesse Douglas has been appointed to an assistant professorship at Brooklyn College.

Associate Professor R. C. Hildner of Mt. Union College has been appointed to an assistant professorship at the College of Wooster.

Professor E. J. Moulton of Northwestern University is on leave of absence and Professor H. T. Davis is acting head of the department of mathematics.

Assistant Professor W. H. Myers has been appointed acting head of the mathematics department at San José State College.

Mr. N. D. Nelson of the University of Wisconsin has been appointed to an assistant professorship at Amherst College.

Dr. E. A. Nordhaus of the University of Wisconsin has been appointed to an assistant professorship at Michigan State College.

Assistant Professor C. V. L. Smith of Lafayette College is now a lieutenant,

E-V (RS), U.S.N.R. and is officer-in-charge of the Naval Training School, Elementary Electricity and Radio Matériel at the University of Houston, Texas.

Associate Professor V. H. Wells of Williams College has been promoted to a full professorship.

Rev. C. R. Wheeler, professor of mathematics at St. Bonaventure College, has been commissioned a first lieutenant in the Army Chaplain Corps.

Professor W. E. Wilson of the Colorado School of Mines has been appointed research engineer with the Armour Research Foundation of Chicago.

Dr. P. M. Young of Miami University is now teaching navigation at the U.S.N.R. Midshipmen's School in New York City.

The following appointments to instructorships are announced:
Allegheny College: R. E. Smith
The University of Chicago: H. L. Meyer
Cornell University: E. J. Scott (part-time)
Eau Claire State Teachers College (Wisconsin): Bjorne R. Ullsvik
Illinois Institute of Technology: Dr. Edward Helly
The University of Michigan: P. S. Jones
The University of North Carolina: Dr. C. M. Smith
Pennsylvania State College: Dr. Helen B. Owens
Tuskegee Institute: Dr. J. E. Wilkins
The University of Wisconsin: Dr. F. A. Butter, Jr.

Assistant Professor L. I. Neikirk of the University of Washington died on December 10, 1942. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS

Recent action by the National Headquarters of the Selective Service System recognizes a critical shortage in the field of mathematics and sets up a new procedure whereby a national point of view may be brought to bear on the decisions of local boards in the cases of registrants who are occupied as mathematicians or are students of mathematics. This recent action of National Headquarters is best understood by reference to the historical background.

On November 7, 1942, National Headquarters announced a policy under which the Director of Selective Service might authorize the appointment of a

National Committee in a critical scientific or specialized field, to assist the Selective Service System by reviewing affidavits for occupational classification; at the same time the National Committee on Physicists was established. On April 5, 1943, this policy was amended so as to apply to mathematicians as well as to physicists; *the National Committee on Physicists was replaced by the National Committee on Physicists and Mathematicians.* In addition to the two physicists who, with representatives of the National Roster of Scientific and Specialized Personnel and of the public, constituted the earlier Committee, the new National Committee on Physicists and Mathematicians will include two mathematicians.

This procedure involves no change in the fundamental principles underlying the granting of occupational deferment, which remains, in the first instance, a matter for the decision of each registrant's local board. The creation of the National Committee on Physicists and Mathematicians does, however, provide a means whereby the individual local boards will be advised concerning the national as well as the local needs in the critical fields of both physics and mathematics.

In the AMERICAN MATHEMATICAL MONTHLY, March, 1943, attention was called to amendments made on December 14, 1942, to Occupational Bulletins No. 10 and No. 23. *The amended Bulletins remain in force and should be consulted in preparing requests for occupational classification of teachers or students of mathematics.**

Mathematicians should note the changes in procedure made necessary by the establishment of the National Committee on Physicists and Mathematicians. *The principal one is that the employer is now to file papers with the National Committee as well as with the employee's local board.* (See 3(b) below.) The following detailed outline, it is hoped, will be of value to mathematicians and their employers as the new procedure goes into effect.

1. Although full responsibility for presenting occupational deferment requests lies with employers,† the mathematician-registrant will be wise to see that his employer is fully informed of the new procedure established by Activity and Occupational Bulletin No. 35 as amended on April 5, 1943.

2. The employer must first decide whether he regards the registrant as a "necessary" man for whom he wishes to request occupational deferment (or continued occupational deferment). In the case of men with dependents, even if deferred at present, it is desirable to file information with the local boards establishing their occupational status as further ground for deferred classification.

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* The substance of the amended Bulletins No. 10 and No. 23 is to be found also in Activity and Occupation Bulletins No. 33-5 and No. 33-6, issued on March 1, 1943.

† Throughout this discussion the term "employer" means that person within an organization—industrial, educational, or governmental—who has been assigned the responsibility for handling deferment problems. In educational institutions, the head of the mathematics department must be an active adviser in matters involving both staff and students.

3. For every registrant for whom occupational deferment (or continued deferment) is to be requested, the employer will:

a) Fill out *completely* Selective Service Form 42 or 42A in triplicate. File a copy with the local board with a brief letter stating that the original has been sent to the National Committee on Physicists and Mathematicians as authorized in Selective Service Activity and Occupation Bulletin No. 35, issued March 1, 1943, and amended April 5, 1943.

b) Send the original and one copy of Form 42 or 42A (see also (c) below) to the:

NATIONAL COMMITTEE ON PHYSICISTS AND MATHEMATICIANS
National Roster of Scientific and Specialized Personnel
Washington, D. C.

c) Send in addition to Form 42 or 42A, a "Report of Status" for the National Committee on Physicists and Mathematicians. Copies of these "Report of Status" blanks may be secured from the National Committee on Physicists and Mathematicians.

d) For those *not now deferred*, this procedure should be followed at once. For those *now deferred*, this procedure should be followed *at least thirty days before the expiration of the deferment period.*

4. For every registrant for whom deferment is not to be requested by his employer or college, a "Report of Status" in duplicate should be filled out and sent to the National Committee on Physicists and Mathematicians at once so that efforts may be made to place him where his talents and training will be of best service to the war effort.

The plan under which deferment applications for mathematicians are now to be made is announced in Activity and Occupation Bulletin No. 35, as amended on April 5, 1943. Extracts giving the essential features of this plan are published herewith:

"*Filing Affidavit—Occupational Classification.* When such a National Committee has been appointed for a scientific or specialized field, the employer or recognized university or college desiring occupational classification for a registrant who possesses the training, qualification, or skill in that field, or is in training or preparation therefor, may prepare Affidavit—Occupational Classification (Form 42 or 42A) in duplicate, file the copy with the local board in the usual manner, and forward the original to the National Committee. (Part I, C, 3)

"*Action by the National Committee.* When the National Committee receives a Form 42 or 42A, it will investigate the registrant. If, in its opinion, the registrant possesses the training, qualification, or skill and is a necessary man in an essential activity or a necessary man in training or preparation therefor, the National Committee is authorized to place a stamped endorsement prescribed by the Director of Selective Service on the original Form 42 or 42A and to file the form with the registrant's local board. (Part I, C, 4)

"*Consideration by local board.* When the original Form 42 or 42A, stamped by the National Committee, is received by the local board, it shall be considered as showing that the registrant has been investigated, that in the opinion of the National Committee he possesses the training, qualification, or skill required, and that, as the case may be, he is a necessary man in an essential activity or a necessary man in training or preparation in such scientific and specialized field. (Part I, C, 5)

"*Classification Advice* (*Form 59*). When a Form 42 or 42A, stamped by a National Committee, has been received for a registrant and the local board, nevertheless, classified the registrant as available for military service or for assignment to work of national importance, the local board shall notify the National Committee, as well as the employer or university or college, by mailing to the National Committee and to the employer or university or college Classification Advice (Form 59). In such case the National Committee, as well as the employer or university or college, may appeal to the board of appeal from classification of the registrant. (Part I, C, 6)

"*National Committees.** The following National Committees for scientific and specialized fields presently are authorized by the Director of Selective Service:

PHYSICISTS AND MATHEMATICIANS

The National Committee on Physicists and Mathematicians, National Roster of Scientific and Specialized Personnel, Washington, D. C.

ENGINEERS AND CHEMISTS

The National Committee on Engineers and Chemists, National Roster of Scientific and Specialized Personnel, Washington, D. C." (Part I, C, 7)

J. R. KLINE,

May 11, 1943. *Secretary of the War Policy Committee*

THE CHICAGO INSTITUTE OF MILITARY STUDIES

The Institute of Military Studies at the University of Chicago was established to advance civilian knowledge of military practice, theory, and history. Classes are conducted for men anticipating military service or for those who expect to train others for service.

The Institute has published under its own imprint three different manuals, namely, *The Organization of the Army*, *A Review of Arithmetic*, prepared by Mr. Zens Smith, and *Some Military Applications of Elementary Mathematics*, published with the permission of The Department of Mathematics, United States Military Academy at West Point. Over nine thousand copies of the last pamphlet have already been distributed, and it is being widely used in public schools, in colleges, and in military and naval training centers.

* As amended 4-5-43.

Individual copies of *A Review of Arithmetic* will be supplied at twenty-five cents. Copies of *Some Military Applications of Elementary Mathematics* are available at the cost price of fifteen cents each. Requests for these pamphlets should be addressed to: The Institute of Military Studies, The University of Chicago, Chicago, Illinois.

Two other texts sponsored by the Institute are now in press and shortly will be available to the public. The first of these will be of especial interest to mathematicians, and is entitled *Manual for Instruction in Military Maps and Aerial Photography*, prepared by Dean Norman F. Maclean, Acting Director of the Institute, and by Mr. Everett C. Olson, Assistant Professor of Geology at the University of Chicago. This manual differs from other manuals on cartography in that, as the title indicates, it is prepared for instructors. In its organization, it recognizes the limited time allotted to map instruction in the training programs of the Army, and it assumes no more than the average background of the inductees who will receive that training; also, consideration is given to the limited facilities and materials available for civilian instructors.

BIBLIOGRAPHY ON ARTILLERY

Professor J. M. Thomas was invited to submit a brief list of titles which might be used for classroom reference upon the mathematics of artillery fire. With the exception of the last title, the following short bibliography is due to him.

H. G. Brohop, *Elements of modern field artillery*, Banta, Menasha, Wisconsin, 1917.

T. J. Hayes, *Elements of ordnance*, Wiley, New York, 1938.

T. J. Hayes, *Exterior ballistics*, Wiley, New York, 1938, $1.00.

J. Haag, *Application au tir*, Gauthier-Villars, Paris, 1926.

F. R. Moulton, *New Methods in exterior ballistics*, University of Chicago Press, Chicago, Ill., 1926.

K. Popoff, *Das Hauptproblem der äusseren Ballistik in Lichte der modernen Mathematik*, Leipzig, Akademische Verlagsgesellschaft, 1932.

W. H. Tschappat, *Ordnance and gunnery*, New York, 1917.

Sophia H. Levy, *Introductory artillery mathematics and antiaircraft mathematics*, University of California Press, Berkeley, Calif., 1943.

Coast artillery, Military Service Publishing Co., Harrisburg, Pa., 1942, $6.00.

Antiaircraft defense, Military Service Publishing Co., Harrisburg, Pa., 1943, $2.00.

Coast artillery field manual, FM 4–15, Government Printing Office, Washington, D. C., 1940, $.50.

Coast artillery field manual, FM 4–10, Government Printing Office, Washington, D. C., 1940, $.25.

Field artillery field manual, FM 6–40, Government Printing Office, Washington, D. C., 1939, $.25.

Orientation, The Coast Artillery School, Fort Monroe, Va., 1942, $.90

Mathematics for the coast artillery officer, The Coast Artillery School, Fort Monroe, Va., 1943, $.25.

Field artillery, gunnery, Book 161, Field Artillery School, Fort Sill, Oklahoma.

J. M. Thomas, *Elementary mathematics in artillery fire*, McGraw-Hill, New York, 1942.

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following thirty-eight persons have been elected to membership on applications duly certified:

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J. F. ARENA, A.M. (Michigan) Teacher, Community High School, Hardin, Ill.

H. C. BOARDMAN, B.S., C.E. (Illinois) Research Engr., Chicago Bridge and Iron Co., Chicago, Ill.

A. C. COHEN, JR., Ph.D. (Michigan) Chief, Quality Control, Picatinny Arsenal, Dover, N. J.

MARGARET R. DAVIS, A.M. (Columbia) Instr., Antioch Coll., Yellow Springs, Ohio

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W. A. FRASER, Ph.D. (Iowa State) Physicist, in charge, Glass Plant Research, Bausch and Lomb Optical Co., Rochester, N. Y.

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W. J. HORN, B.S. in Archt. (A. and M. Coll. of Texas) Structural Engr., Truscon Steel Co., Houston, Texas

FREE JAMISON, Ph.D. (Pittsburgh) U.S.N., Air Navigation School, Hollywood, Fla.

W. J. LEVEGUE. Student, Univ. of Colorado, Boulder, Colo.

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ELLA MARTH, M.S. (St. Louis Univ.) Teacher, Eliot School, St. Louis, Mo.

W. T. MARTIN, Ph.D. (Illinois) Asst. Prof., Massachusetts Inst. of Tech., Cambridge, Mass.

W. H. MILLS, A.B. (Swarthmore) Junior Physicist, Ballistic Research Lab., Aberdeen Proving Ground, Md.

ALFONSO NÁPOLES, Ph.D. (National Univ., Mexico) Prof., Head of Dept., National Univ., Mexico, D. F.

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MAX PETERS, M.S. (C.C.N.Y.) Teacher, New Utrecht High School, Brooklyn, N. Y.

GEORGE POLYA, Ph.D. (Budapest) Asso. Prof., Stanford Univ., Stanford University, Calif.

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O. T. SCHULTZ, M.S. (Chicago) Asst. Project Engr., Sperry Gyroscope Co., Inc., Garden City, N. Y.

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L. A. WALKER, A.M. (Stanford) Teacher, Math. and Physics, High School, San Mateo, Calif.

J. T. WIEGAND, B.E. (Chicago T.C.) Instr., Illinois Inst. of Tech., Chicago, Ill.

R. S. WOLFE, A.M. (Washington) Instr., Northwestern Univ., Evanston, Ill.

W. B. CARVER, *Secretary-Treasurer*

SPRING MEETING OF THE MICHIGAN SECTION

The nineteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 14, 1942. Morning and afternoon sessions and a business-luncheon meeting were held, at all of which the chairman of the Section, Professor T. R. Running, presided.

The meeting was attended by fifty-six persons, including the following thirty-four members of the Association: H. M. Ackley, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, W. M. Borgman, J. B. Brandeberry, R. V. Churchill, C. J. Coe, C. C. Craig, Wayne Dancer, C. M. Erikson, K. W. Folley, J. F. Heyda, T. H. Hildebrandt, Frances C. Hinds, L. A. Hopkins. L. S. Johnston, P. S. Jones, Wilfred Kaplan, W. C. Krathwohl, Theodore Lindquist, Sister Mary Paula, E. D. McCarthy, A. L. Nelson, J. K. Peterson, H. H. Pixley, G. Y. Rainich, E. D. Rainville, L. J. Rouse, T. R. Running, E. R. Stabler, A. G. Swanson, Fern Welker.

At the business meeting Professor Wayne Dancer of the University of Toledo was elected chairman for 1942–43, and Professor C. J. Coe of the University of Michigan was reelected secretary-treasurer.

At the morning session the following program of seven papers was presented, but due to the dangerously icy condition of the roads, Professors Stewart and Sleight were unable to be present and their papers were read by title.

1. *Boolean algebra as an introduction to postulational methods* by Professor E. R. Stabler, University of Michigan.

Professor Stabler showed how a set of postulates and theorems for Boolean algebra may be used for the study of consistency, independence, and categoricalness of a set of postulates. The proposed postulates were those listed as Set III, pp. 20–28, this MONTHLY, January, 1941, together with the requirement of a unit element. The speaker also indicated that the system could conveniently be used to study such concepts as the relative truth of theorems and the equivalence of different sets of postulates.

2. *Discontinuous rates of change* by Professor E. D. McCarthy, University of Detroit.

The speaker started with the assumption that time varies discontinuously, taking on only integral values. Thus velocity and distance in rectilinear motion are given by $V=\sum a$ and $S=\sum V$ respectively. Several simple cases were in-

vestigated, and results were obtained similar to those for the continuous case. In particular, a study of the case $a = kS$ led to the difference equation

$$V_n - (k + 2)V_{n-1} + V_{n-2} = 0,$$

of which the solution is $V = c_1L^t + c_2L^{-t}$, where L is a root of $L^2 - (k+2)L + 1 = 0$. The exponential and trigonometric cases were discussed.

3. *A pair of linear transformations* by Professor G. Y. Rainich, University of Michigan.

Professor Rainich discussed a pair of linear transformations in the plane. He showed that in addition to satisfying the Hamilton-Cayley equations individually, they also satisfy a third equation involving both of them. A condition for the existence of a common invariant direction was obtained, and expressed in terms of the coefficients of the specified equations.

4. *A maximum problem* by Dr. B. M. Stewart, Michigan State College.

Dr. Stewart's paper was published in this MONTHLY vol. 49, 1942, pp. 454–456.

5. *Arithmetic according to Cocker* by Professor E. R. Sleight, Albion College.

This paper gave a historical account of Cocker himself, and presented a review of the arithmetics which were purported to have been written by him. Cocker's arithmetics were compared with those which preceded them, and it was pointed out that his predecessors seemed to know when to stop, but that Cocker continued indefinitely.

6. *Solution of difference equations by the Mellin inversion theory* by Dr. E. W. Paxson, Wayne University, introduced by Professor Nelson.

Dr. Paxson discussed the solution of linear difference equations with constant coefficients. He reduced the problem to that of solving certain integral equations. The integral equations were solved by means of the Mellin inversion theory.

7. *Demonstration of mathematical principles and applications with moving apparatus* by P. S. Jones, Edison Institute.

Mr. Jones presented a demonstration of still and moving mathematical models constructed by his students at the Edison Institute. The models displayed a professional finish due to the mechanical expertness of his classes, but the speaker pointed out that the construction of some of the simpler ones was within the ability of the average class, and could form a valuable teaching aid. The following models were demonstrated:

(1) Pin and string construction of the ellipse; a pair of rolling ellipses; four bar linkage for the same motion.

(2) Elliptic trammel for blackboard construction.

(3) Elliptic billiard table.

(4) String models of the conics.

(5) Free hanging and loaded cables.

(6) Rolling wheel tracing a cycloid.

(7) Cycloidal and linear tracks showing the cycloid as tautochrone and brachistochrone.

(8) Involute of a circle and involute gear.

(9) Spiral of Archimedes and uniform motion cam.

(10) Limaçon and simple harmonic motion.

(11) Trisectrix of Archimedes.

At the afternoon session the regional governor, Professor W. C. Krathwohl of the Illinois Institute of Technology, presented an invited address on "Aptitude tests as aids to administrators and teachers of mathematics." Professor Krathwohl illustrated his remarks with tests given at the Illinois Institute of Technology. He showed how these tests are analyzed and used in the guidance of students at the Institute, and in placing them in positions later. He made it clear that such tests, when properly devised and administered, can be of great value to the educational institution, to the student, and to the employer.

At the luncheon meeting Professor Krathwohl discussed the general affairs of the Association, and reported on the September meeting of the Board of Governors.

C. J. COE, *Secretary*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pa., April, 1944
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN
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Fifth edition, June 1941, ii, 76 pages

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In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association. Through two subsequent gifts the prize is now awarded every three years. The last award was made in December 1941 to Professor Saunders Mac Lane for his two papers in the *American Mathematical Monthly:* "Modular fields," volume 47, 1940, pp. 259-274 and "Some recent advances in algebra," volume 46, 1939, pp. 3-19.

As determined more recently by the Trustees, the prize is to be fifty dollars and is to be awarded for a noteworthy expository paper such as will come within the range of profitable reading of members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included; they carry their own reward.

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CONTENTS

Earle Raymond Hedrick . . . . . W. B. FORD 409
Mathematics, 400 B.C.–300 B.C. . . . . MAX DEHN 411
Applied Mathematics and the Present Crisis. R. G. D. RICHARDSON 415
Geodesics and Plane Arcs on an Oblate Spheroid . . L. E. WARD 423
Quadratic Forms with Linear Constraints . . . . H. B. MANN 430
Discussions and Notes . . . . . .
. J. B. SHAW, L. S. RECHT, W. R. RANSOM and J. V. BREAKWELL 434
Recent Publications 438
Problems and Solutions. 454
News and Notices 461
War Information 464
The United States Armed Forces Institute 464
The Michigan Division for Emergency Training 464
Pre-Induction Courses in Mathematics 465
From the War Policy Committee 466
The Army Specialized Training Program. 466
Women in Aerology and Electronics 470
The Mathematical Association of America 471
Ohio Section Meeting 471
Missouri Section Meeting 473
Meetings of the Association and its Sections. 474

AUGUST-SEPTEMBER 1943